MATH601 Spring 2008
Exam 7 Solutions

1. Consider the position $(7, 11, 13, 15)$ in Nim. Find all the winning moves.

   We have $\star 7 + \star 11 + \star 13 + \star 15 = (\star 1 + \star 2 + \star 4) + (\star 1 + \star 2 + \star 8) + (\star 1 + \star 4 + \star 8) + (\star 1 + \star 2 + \star 4 + \star 8) = \star 2 + \star 4 + \star 8 = \star 14$. Hence the position is of value $\star 14$.

   We now calculate $\star 14 + \star 7 = (\star 2 + \star 4 + \star 8) + (\star 1 + \star 2 + \star 4) = \star 1 + \star 8 = \star 9$.
   No winning move is possible from this pile, as $9 > 7$.

   We now calculate $\star 14 + \star 11 = (\star 2 + \star 4 + \star 8) + (\star 1 + \star 2 + \star 8) = \star 1 + \star 4 = \star 5$.
   The only winning move from this pile is to remove 6, leaving 5.

   We now calculate $\star 14 + \star 13 = (\star 2 + \star 4 + \star 8) + (\star 1 + \star 4 + \star 8) = \star 1 + \star 2 = \star 3$.
   The only winning move from this pile is to remove 10, leaving 3.

   We now calculate $\star 14 + \star 15 = (\star 2 + \star 4 + \star 8) + (\star 1 + \star 2 + \star 4 + \star 8) = \star 1$.
   The only winning move from this pile is to remove 14, leaving 1.

2. Consider the Hackenbush game consisting if a single stalk of length n, where the edges are alternately labeled (starting from the ground) $L, R, L, R, \ldots$. Let $f(n)$ denote the value of this game. For example, $f(0) = 0, f(1) = 1, f(2) = 1/2$.
   Prove that these values are ordered as $f(0) < f(2) < f(4) < \cdots < f(5) < f(3) < f(1)$.

   We prove this by induction on n; the base cases $n = 1, 2$ were done already. The key observation is that when n is even the stalk ends[1] in R, whereas when n is odd the stalk ends in L. Hence, every move L makes will yield a stalk of even length, whereas every move R makes will yield a stalk of odd length. If we use $even(n)$ to denote the largest even integer strictly less than n, and $odd(n)$ to denote the largest odd integer strictly less than n, we have:
   $f(n) = \prec f(0), f(2), f(4), \ldots, f(even(n)) | f(1), f(3), f(5), \ldots, f(odd(n)) \succ$.
   Hence, by the seniority principle, $f(n)$ is strictly greater than $f(m)$ for every m that is even and less than n. Also, $f(n)$ is strictly less than $f(m)$ for every m that is odd and less than m.

   In fact, it turns out that $f(n) = \begin{cases} 2/3(1 - (1/2)^n) & n \text{ even} \\ 2/3(1 + (1/2)^n) & n \text{ odd} \end{cases}$ hence $\lim_{n\to\infty} f(n) = 2/3$.

3. Exam grades: 96, 96, 95, 88, 87, 85, 80, 79, 72

4. Survey results: Naturals > Game Theory > Hyperreals > Surreals > Ordinals > Cardinals > Reals
   Perhaps not surprisingly, this is exactly the order of your exam averages.

[1] That is, the topmost edge, furthest away from the ground.