MATH601 Spring 2008
Exam 2 Solutions

1. Carefully and precisely state the definition of real numbers that we have been using.

   $x = \prec L|R \succ$ is a real number precisely when L, R are each subsets of the rationals $\mathbb{Q}$, such that the following three axioms are satisfied:

   (a) L is nonempty, and possesses neither greatest nor least element. Same for R.

   (b) Every element of L is less than every element of R.

   (c) At most one element of $\mathbb{Q}$ is missing from $L \cup R$.

   Note: Some of you mentioned that L is "less than" or "to the left of" x. These are certainly correct, and helpful for your intuition, but they may not be part of a definition. The definition only contains rational numbers; this intuitive understanding requires having $<$ defined between rational numbers and the very real number we are trying to define. At best this is circular logic.

2. Prove that for every real number x with $x \geq 0$, that $0 \times x = 0$.

   For convenience, set $y = 0 \times x$. The definition of multiplication tells us that $R(y) = \{x^R \times 0^R\}$, and that $L(y) = L(0) \cup \{x^L \times 0^L | x^L \geq 0, 0^L \geq 0\}$; however there are no elements of $L(0)$ that are nonnegative; hence $\{x^L \times 0^L | x^L \geq 0, 0^L \geq 0\} = \emptyset$, and thus $L(y) = L(0)$.

   There are several ways to show that $y = 0$. One way is to show $y \leq 0$ and $0 \leq y$. Because $y^L \in L(0)$, we must have $y^L < 0^R$ (since 0 is a real number, apply the second property of real numbers). This proves that $y \leq 0$. Now take $y^R = x^R \times 0^R$. Because $x \geq 0$, we have $x^R > 0^L$, so every x^R is bigger than every negative fraction, hence is a nonnegative fraction. Because 0 is a real number, $0^R > 0^L$, so again every 0^R is a nonnegative fraction. Putting these together, since $y^R = x^R \times 0^R$, every y^R is the product of two nonnegative fractions, hence itself is a nonnegative fraction, so $y^R > 0^L$. This proves that $0 \leq y$, and hence by definition of $=$ that $y = 0$.

   ALTERNATE APPROACH: we can show that $y, 0$ have the same L, R sets, and therefore are identical. First, we show that every $y^R > 0$. Because $x \geq 0$, we must have $x^R > 0^L$, hence $x^R \geq 0$ (since it is greater than every negative rational). But we cannot have $0 \in R(x)$: since $R(x)$ has no least element there would therefore be some element smaller than 0, violating $x^R > 0^L$. Hence $x^R > 0$; but also $0^R > 0$, so their product is positive. Hence $R(y) \subseteq R(0)$. Finally, we show that $R(0) \subseteq R(y)$. Let $a \in R(0)$, and $b \in R(x)$. Set $c = a/b$. We have $c \in R(0)$, so $c \times b \in R(y)$; hence $a \in R(y)$. So $R(0) \subseteq R(y)$.

3. Exam grades: 87, 82, 82, 78, 78, 75, 75, 66, 61