MATH 579 Spring 2009 Supplement: Basic Enumeration

Consider the set of numbers $1, 2, \ldots, n$, denoted as $[n]$. We frequently model counting problems using $[n]$.

A **set** drawn from $[n]$, denoted as $\{1, 2, 3\}$, is a collection of distinct objects where order does not matter. $\{1, 2, 3\} = \{1, 3, 2\}$, and $\{1, 2, 2\}$ is not a set. $\{1, 2, 3\}$ contains 3 elements.

A **multiset** drawn from $[n]$, denoted as $\{1^2, 2^1, 3^5\}$, is a collection of not necessarily distinct objects where order does not matter. $\{1^2, 2^1, 3^5\}$ has two 1's, one 2, five 3's, and no other numbers. It contains eight elements. $\{1^2, 2^1, 3^5\} = \{1^2, 3^5, 2^1\} \neq \{1^2, 2^2, 3^5\}$. For convenience, we can think of $\{1, 2, 2\}$ (not really a multiset) as $\{1^1, 2^2\}$.

A **list** or **tuple** drawn from $[n]$, denoted as $(1, 2, 3)$, is a collection of not necessarily distinct objects where order matters. $(1, 2, 3) \neq (1, 3, 2)$. $(1, 2, 2)$ is also list; all of these are lists of three elements, or 3-tuples. Sometimes it is convenient to think of lists as **words** over the **alphabet** $[n]$, in which case we write $123, 132, 122$ as 3-letter words.

The fourth possibility is a collection of distinct objects where order matters. Unfortunately there is no standard terminology or notation in this case, so we will use the awkward **list of distinct elements (l.o.d.e.)** and use the same $(1, 2, 3)$ as for lists.

Before we count our objects (sets/multisets/lists/l.o.d.e's), we need to ask how many times we can use each element of $[n]$ in each of the objects we are counting. Must we use each element at least once? It is often very helpful in modeling to write down several examples of the objects we are counting, to determine which type of object we want, and whether there are restrictions on the use of elements from $[n]$.

Basic solution: no restrictions on use of $[n]$.
There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ sets of size k, for $k \in [0, n]$ (none for k outside this range) "binomial coefficients". There are $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$ multisets of size k. There are n^k k-tuples. There are $(n)_k = \frac{n!}{(n-k)!}$ k-lists of distinct elements "falling powers". These are all from Chapter 3 and are summarized in Table 3.1.

Each element of $[n]$ must appear at least once.
This is silly for sets – there is exactly one such set of size k if $k = n$, and none if $k \neq n$. For l.o.d.e.'s, there are $n!$ of size n, and none of any other size. Your book calls these 'permutations'; they are in Table 3.1. There are $\left(\binom{n}{k-n}\right) = \binom{k-1}{k-n} = \binom{k-1}{n-1}$ multisets of size k. Your book calls these 'compositions'. There are $n!S(k, n)$ lists of size k, where $S(k, n)$ denotes the Stirling number of the second kind. Your book calls these 'surjections'. Compositions and surjections appear in Table 5.1.

Various other possibilities are possible, many of which are sums of some of the previous cases. One important special case is if each element from $[n]$ has a specified number of times it must appear. These are called multinomial coefficients, and appear in Theorem 3.5 and Table 3.1.

An important connection is between these objects (sets, multisets, lists, l.o.d.e's) and functions. A 3-tuple $(1, 3, 4)$ can be thought of as a function f on the set {first, second, third}, where $f(\text{first}) = 1, f(\text{second}) = 3, f(\text{third}) = 4$. $(1, 3, 4)$ completely determines this function f. The function's domain is {first, second, third}, and its range is $[4]$. $(1, 3, 3)$ is a different function.

These functions are *injective* (one-to-one) precisely when the objects can have an element at most once. That is, sets and l.o.d.e's are injective, while multisets and lists are not necessarily injective. These functions are *surjective* (onto) precisely when each element must appear at least once.

The domain has distinct elements precisely when the objects are ordered. The list $(1, 3, 4)$ has $f(\text{first}) = 1, f(\text{second}) = 3, f(\text{third}) = 4$. The set $\{1, 3, 4\}$ has $f(\text{something}) = 1, f(\text{something}) = 3, f(\text{something}) = 4$. There are three somethings in the domain, but we can't tell them apart, since $\{1, 3, 4\} = \{1, 4, 3\}$. Hence sets and multisets have indistinct domains, and lists and l.o.d.e's have distinct domains. In all the above cases, the range has distinct elements; otherwise the set/multiset/list/l.o.d.e. analogy breaks down.