

## Math 579 Exam 9 Solutions

1. Solve the following recurrence.  $a_n = 6a_{n-1} - 9a_{n-2}$ ,  $a_0 = 0$ ,  $a_1 = 1$ .

This homogeneous recurrence has characteristic equation  $x^2 - 6x + 9 = 0$ , which has a double root of  $x = 3$ . Hence, the general solution is  $a_n = \alpha 3^n + \beta n 3^n$ . The initial conditions give us  $0 = a_0 = \alpha + 0$ ,  $1 = a_1 = \alpha 3 + \beta 3$ , which has solution  $\alpha = 0$ ,  $\beta = 1/3$ . Hence, the solution is  $a_n = n 3^{n-1}$ .

2. Solve the following recurrence.  $a_n = 3a_{n-1} + 2n + 1$ ,  $a_0 = 0$ .

The homogeneous version has general solution  $a_n = \alpha 3^n$ . For the nonhomogeneous version, try  $a_n = An + B$ . We have  $An + B = 3(A(n-1) + B) + 2n + 1$ . We rearrange to  $n(3A + 2 - A) + (-3A + 3B + 1 - B) = 0$ ; hence  $3A + 2 - A = 0$ ,  $-3A + 3B + 1 - B = 0$ . This has solution  $A = -1$ ,  $B = -2$ . Hence the general nonhomogeneous solution is  $a_n = \alpha 3^n - n - 2$ . We apply the initial condition to get  $0 = a_0 = \alpha 3^0 - 0 - 2$ . Hence  $\alpha = 2$  and the solution is  $a_n = 2(3^n) - n - 2$ .

3. A gambler repeatedly plays a game against a casino, until one of them runs out of money. Each time, the gambler has probability  $s$  of nothing happening, probability  $p$  of winning \$1, and probability  $q$  of losing \$1, with  $s + p + q = 1$ . The gambler starts with  $n$  dollars, and the casino with  $m - n$  dollars. What is the probability that the gambler will run out of money before the casino?

This is similar to Example 4 in the handout. The conditions give us the recurrence relation  $a_n = sa_n + pa_{n+1} + qa_{n-1}$ , with characteristic equation  $px^2 + (s-1)x + q = 0$ . However, this has (surprisingly) the same roots as before, namely 1 and  $q/p$ . Therefore, the rest of Example 4 applies verbatim, giving solutions  $1 - \frac{1-(q/p)^n}{1-(q/p)^m}$  for  $p \neq q$  and  $1 - (n/m)$  for  $p = q$ .

4. Solve the following recurrence.  $a_n = a_{n-1} + a_{n-2} - a_{n-3} + 2$ ,  $a_0 = 4$ ,  $a_1 = 0$ ,  $a_2 = 5$ .

The characteristic equation of the homogeneous recurrence is  $x^3 - x^2 - x + 1 = 0$ , which factors as  $(x-1)^2(x+1) = 0$ . Hence the general homogeneous solution is  $a_n = \alpha + \beta n + \gamma(-1)^n$ . For the particular nonhomogeneous solution, we note that polynomials of degree 0 and 1 are already represented in the homogeneous solution set. Hence we try  $a_n = An^2$ .  $An^2 = A(n-1)^2 + A(n-2)^2 - A(n-3)^2 + 2 = A(n^2 - 2n + 1) + A(n^2 - 4n + 4) - A(n^2 - 6n + 9) + 2 = An^2 - 4A + 2$ . Hence  $A = 1/2$ , and the general nonhomogeneous solution is  $a_n = \alpha + \beta n + \gamma(-1)^n + n^2/2$ . We apply our initial conditions to get  $4 = a_0 = \alpha + \gamma$ ,  $0 = a_1 = \alpha + \beta - \gamma + \frac{1}{2}$ ,  $5 = a_2 = \alpha + 2\beta + \gamma + 2$ . This has solution  $\alpha = \gamma = 2$ ,  $\beta = -1/2$ , and thus the solution is  $a_n = 2 + 2(-1)^n + (n^2 - n)/2$ .

5. Let  $a_n$  represent the maximum number of regions we can divide the plane into with  $n$  lines. Find and solve a recurrence for  $a_n$ .

We note that  $a_1 = 2$ ,  $a_2 = 4$ . Consider adding the  $n^{\text{th}}$  line. It can cross at most  $n - 1$  lines (each of the previous lines once). Hence, it can cross at most  $n$  regions (since

to go from one region to another it must cross a line). Since it crosses at most  $n$  regions, it can create at most  $n$  new regions. Conversely, we can ensure that it DOES create  $n$  new regions, by making it not parallel to any of the other lines and not pass through any intersection point. Hence  $a_n = a_{n-1} + n$ . This has homogeneous solution  $a_n = \alpha(1)^n = \alpha$ . For the nonhomogeneous version, we guess  $a_n = An^2 + Bn$ . We have  $An^2 + Bn = A(n-1)^2 + B(n-1) + n$ . We solve to find  $A = B = 1/2$ . Hence  $a_n = \alpha + (n + n^2)/2$  is the general nonhomogeneous solution. Our initial condition gives us  $2 = a_1 = \alpha + (1 + 1^2)/2$ , so  $\alpha = 1$  and the solution is  $a_n = (n^2 + n + 2)/2$ .

Part II. Let  $a_0 = A, a_1 = B, a_n = \frac{1+a_{n-1}}{a_{n-2}}$  for  $n > 1$ . Assume that  $a_n \neq 0$  for all  $n$ , so we never divide by zero. Calculate and simplify  $a_2, a_3, a_4, a_5, a_6$ . Look for a pattern, and use it to find  $a_n$ .

$a_2 = \frac{1+B}{A}, a_3 = \frac{A+B+1}{AB}, a_4 = \frac{A+1}{B}, a_5 = A, a_6 = B$ , and then it all begins again (this sequence is periodic).

$$\text{Hence } a_n = \begin{cases} A & n = 5k, k \in \mathbb{N} \\ B & n = 5k + 1, k \in \mathbb{N} \\ (1 + B)/A & n = 5k + 2, k \in \mathbb{N} \\ (1 + A + B)/AB & n = 5k + 3, k \in \mathbb{N} \\ (1 + A)/B & n = 5k + 4, k \in \mathbb{N} \end{cases}$$

Exam statistics: Low grade=32(D); Median grade=41(B); High grade=49(A)