**Math 524 Final Exam 11: 12/16/8**

Please read the exam instructions.

Notes, books, papers, calculators and electronic aids are all forbidden for this exam. Please write your answers on **separate paper**, indicate clearly what work goes with which problem, and put your name on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Keep this list of problems for your records. Show all necessary work in your solutions; if you are unsure, show it. Each problem is worth 10 points. You have approximately 120 minutes.

1. Carefully define the term "vector space".

2. Carefully define the term "complex inner product".

3. Carefully define the term "power vector" (generalized eigenvector).

4. Carefully state Thm 3.7, the Dimension Theorem.

5. Carefully state Thm 7.2, concerning representation of the adjoint of an operator.

6. Solve the system of difference equations $\left\{ \begin{array}{l} \text{x(n)=2y(n-1)} \\ \text{y(n)=3x(n-1)+y(n-1)} \end{array} \right\}$ with $x(0) = 1, y(0) = 0$.

For the next two problems, let $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$.

7. Find all eigenvalues of $A$; give a basis for each eigenspace. HINT: column sums

8. Find the kernel and image of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by multiplication by $A$; that is, $L(x) = Ax$. Is $L$ one-to-one? onto?

For the next four problems, consider the vector space $\mathbb{R}_2[t]$, real polynomials of degree at most 2, with the real inner product $\langle f|g\rangle = \int_0^1 f(t)g(t)dt$. Set $u(t) = t - 1, v(t) = t^2 - 1$, and $V = \text{Span}(u, v) = \left\{at^2 + bt - (a + b)\right\} = \{p(t) : p(1) = 0\}$.

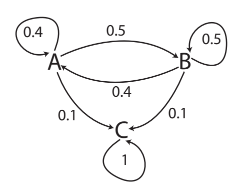
9. Pick any $w \notin V$, and set $B = \{u, v, w\}$. Prove that $B$ is a basis for $\mathbb{R}_2[t]$, and calculate $[1+2t+3t^2]_B$.

10. Let $W = \{at : a \in \mathbb{R}\}$. Prove that $\mathbb{R}_2[t]$ is the internal direct sum of $V, W$.

11. Let $L$ be the linear operator that projects onto $V$. Find a representation of the adjoint $[L^\dagger]_B$, for $B$ a basis of your choice. Is $L$ symmetric? orthogonal?

12. Let $B = \{u, v\}$, a basis for $V$. Calculate two bases for $V^\star$ by specifying their action on each element of $V$. (1) the dual basis $\{\phi_1, \phi_2\}$, (2) the bra basis $\{\langle u|, \langle v|\}$.

13. Consider the Markov chain pictured at right. If the initial distribution is starting in A, i.e. $(1, 0, 0)^T$, find (approximately) the distribution after 12 time steps. You may use the approximation that $(0.9)^{12} \approx 2/7$.




14. Find a linear operator, on the vector space of your choice, that has two eigenvalues: $\lambda = 3$, with algebraic multiplicity 5 and geometric multiplicity 3, and $\lambda = 4$, with algebraic multiplicity 3 and geometric multiplicity 1.

15. Find all $2 \times 2$ complex matrices that are simultaneously anti-symmetric and unitary.

16. (extra credit) Prove that every probability matrix has eigenvalue $\lambda = 1$.