Math 524 Final Exam Solutions

1. Carefully define the term "vector space".

   A vector space is a set of vectors, a field of scalars, and two operations – vector addition and scalar multiplication – that are each closed. There are eight axioms these operations must satisfy (each for all vectors x, y, z and all scalars a, b): $x + y = y + x$, $(x + y) + z = x + (y + z)$, $\exists 0$ with $0 + x = x$, $\forall x \exists(-x)$ with $x + (-x) = 0$, $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $1x = x$, $(ab)x = a(bx)$

2. Carefully define the term "complex inner product".

   A complex inner product is a function from $\mathbb{C}^n$ to $\mathbb{C}$ satisfying five axioms (each for all vectors x, y, z and all scalars a, b): $\langle ax + by|z\rangle = \bar{a}\langle x|z\rangle + \bar{b}\langle y|z\rangle$, $\langle x|ay + bz\rangle = a\langle x|y\rangle + b\langle x|z\rangle$, $\langle x|y\rangle = \overline{\langle y|x\rangle}$, $\langle x|x\rangle$ is real and positive for $x \neq 0$, $\langle 0|0\rangle = 0$.

3. Carefully define the term "power vector" (generalized eigenvector).

   Let L be a linear operator (or matrix). A nonzero vector ξ is a power vector if $(L - \lambda I)^p \xi = 0$, for some eigenvalue λ and some positive integer p.

4. Carefully state Thm 3.7, the Dimension Theorem.

   Given any vector spaces V, W, and any linear transformation $L : V \to W$, the dimension of V equals the dimension of the kernel of L, plus the dimension of the range of L.

   Note: technically, this theorem only holds if the three vector spaces involved *have* dimensions – this may require the axiom of choice.

5. Carefully state Thm 7.2, concerning representation of the adjoint of an operator.

   Given a finite dimensional inner product space V, an orthonormal basis B, and a linear operator L, we have $[L^\dagger]_B = \overline{[L]_B^T}$. Alternatively, using bras and kets, $L^\dagger = \sum_{i,j=1}^n |b_i\rangle\langle Lb_i|b_j\rangle\langle b_j|$.

6. Solve the system of difference equations $\left\{ \begin{array}{l} \text{x(n)=2y(n-1)} \\ \text{y(n)=3x(n-1)+y(n-1)} \end{array} \right\}$ with $x(0) = 1, y(0) = 0$.

   We have $\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x(n-1) \\ y(n-1) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}^n \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The matrix has eigenvalues $-2, 3$ with eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (respectively). Set $P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$; $P^{-1} = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}$, and $D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. Hence $\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} = PDP^{-1}$ and $\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}^n = PD^nP^{-1} = 1/5 \begin{pmatrix} 3(-2)^n + 2 \cdot 3^n & (-2)(-2)^n + 2 \cdot 3^n \\ -3(-2)^n + 3 \cdot 3^n & 2(-2)^n + 3 \cdot 3^n \end{pmatrix}$. Hence $x(n) = (3(-2)^n + 2 \cdot 3^n)/5, y(n) = (-3(-2)^n + 3 \cdot 3^n)/5$.

For the next two problems, let $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$.

7. Find all eigenvalues of A; give a basis for each eigenspace. HINT: column sums

   The hint tells us that $\lambda = 2$ is one eigenvalue. The determinant is 8, the trace is 6, hence the other eigenvalues multiply to 4 ($= 8/2$) and add to 4 ($= 6-2$); we conclude that $\lambda = 2$ is the only eigenvalue, with (algebraic) multiplicity 3. We calculate $A - 2I = B = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore x_2, x_3 are free (2-dimensional eigenspace), and $x_1 = -x_2 - x_3$. One basis for E_2 is $\{(-1,1,0)^T, (-1,0,1)^T\}$.

8. Find the kernel and image of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by multiplication by A; that is, $L(x) = Ax$. Is L one-to-one? onto?

   We use elementary row operations to put A into row canonical form; we get I. Hence the rank of A is 3, and the nullity of A is 0. Hence the kernel of L is $\{0\}$, which makes L one-to-one. Because A is rank 3, the image of L is all of $\mathbb{R}^3$, hence L is onto.

For the next four problems, consider the vector space $\mathbb{R}_2[t]$, real polynomials of degree at most 2, with the real inner product $\langle f|g\rangle = \int_0^1 f(t)g(t)dt$. Set $u(t) = t-1, v(t) = t^2 - 1$, and $V = \mathrm{Span}(u,v) = \{at^2 + bt - (a+b)\} = \{p(t) : p(1) = 0\}$.

9. Pick any $w \notin V$, and set $B = \{u, v, w\}$. Prove that B is a basis for $\mathbb{R}_2[t]$, and calculate $[1 + 2t + 3t^2]_B$.

   Many solutions are possible, depending on choice of w. $w(t)$ may be any polynomial, of degree at most 2, such that $w(1) \neq 0$. For example, $w(t) = t$. To prove this is a basis, calculate the representation of B in the standard basis $E = \{1, t, t^2\}$: $\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Using elementary row operations, we find the row canonical form to be I; hence this has column rank 3, and B is independent. Since $|B| = 3$, the dimension of $\mathbb{R}^3$, B is a basis. We can find $[1 + 2t + 3t^2]_B$ using a change-of-basis matrix, or in an ad-hoc manner since B has a nice form: $1 + 2t + 3t^2 = a(t-1) + b(t^2 - 1) + ct$, hence $b = 3, -a - b = 1, a + c = 2$, so $[1 + 2t + 3t^2]_B = [-4, 3, 6]^T$.

10. Let $W = \{at : a \in \mathbb{R}\}$. Prove that $\mathbb{R}_2[t]$ is the internal direct sum of V, W.

    This is a consequence of Theorem 2.13. V, W are subspaces of $\mathbb{R}_2[t]$. To complete the proof, we need to show two things:

    (a) The dimension of V (known to be 2 since u, v are independent), plus the (unknown) dimension of W, equals the dimension of $\mathbb{R}_2[t]$ (known to be 3).

(b) $V \cap W = \{0\}$.

We prove that W is one-dimensional (1) by observing that every polynomial in W is a scalar multiple of every other; hence an independent set can have only one vector in it. We next note that for $f(t) = at$, an element of W, $f(1) = a$. Hence for this to be in V we must have $a = 0$; in this case $f(t) = 0$ which is the zero polynomial (zero vector). This proves (2).

Note: W is not $V^\perp$; there are many subspaces S with $\mathbb{R}_2[t]$ an internal direct sum of V, S.

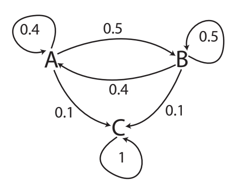
11. Let L be the linear operator that projects onto V. Find a representation of the adjoint $[L^\dagger]_B$, for B a basis of your choice. Is L symmetric? orthogonal?

The best possible basis $B = \{b_1, b_2, b_3\}$ is one that is orthonormal, with b_1, b_2 a basis for V, and b_3 a basis for W. We could find this using Gram-Schmidt on the basis from problem 9, but it is actually not necessary to find B. $L(b_1) = b_1, L(b_2) = b_2, L(b_3) = 0$, hence $[L]_B = ([L(b_1)]_B [L(b_2)]_B [L(b_3)]_B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence $[L^\dagger]_B = [L]_B^T = [L]_B$. This is symmetric, but not orthogonal (it's not invertible, it has 0 as an eigenvalue).

12. Let $B = \{u, v\}$, a basis for V. Calculate two bases for $V^\star$ by specifying their action on each element of V. (1) the dual basis $\{\phi_1, \phi_2\}$, (2) the bra basis $\{\langle u|, \langle v|\}$.

We have $x(t) = at^2 + bt - (a+b) = au + bv$. Hence $[x]_B = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and $\phi_1(x) = a, \phi_2(x) = b$. $\langle u|u\rangle = \int_0^1 (t-1)^2 dt = \int_0^1 t^2 - 2t + 1 dt = 1/3$. $\langle v|v\rangle = 8/15$. $\langle u|v\rangle = \langle v|u\rangle = 5/12$. Hence $\langle u|x\rangle = \langle u|au + bv\rangle = a\langle u|u\rangle + b\langle u|v\rangle = a/3 + 5b/12 = \frac{4a+5b}{12}$. Similarly, $\langle v|x\rangle = \langle v|au + bv\rangle = a\langle v|u\rangle + b\langle v|v\rangle = 5a/12 + 8b/15$.

13. Consider the Markov chain pictured at right. If the initial distribution is starting in A, i.e. $(1, 0, 0)^T$, find (approximately) the distribution after 12 time steps. You may use the approximation that $(0.9)^{12} \approx 2/7$.



This has transition matrix $M = \begin{pmatrix} 0.4 & 0.4 & 0 \\ 0.5 & 0.5 & 0 \\ 0.1 & 0.1 & 1 \end{pmatrix}$. This has known eigenvalue 1, since the column sums are 1. The trace is 1.9, the determinant is 0, so the remaining two eigenvalues have sum 0.9 and product 0; hence the three eigenvalues are $0, 0.9, 1$, with eigenvectors $(-1, 1, 0), (4, 5, -9), (0, 0, 1)$ respectively. Hence, the general solution is $\alpha(1)^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta(0.9)^n \begin{pmatrix} 4 \\ 5 \\ -9 \end{pmatrix} + \gamma(0)^n \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. The initial condition corresponds to $\alpha = 1; \beta = 1/9; \gamma = -5/9$. Evaluating at $n = 12$ gives the approximate solution $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1/9)(2/7) \begin{pmatrix} 4 \\ 5 \\ -9 \end{pmatrix} = \begin{pmatrix} 8/63 \\ 10/63 \\ 45/63 \end{pmatrix}$

14. Find a linear operator, on the vector space of your choice, that has two eigenvalues: $\lambda = 3$, with algebraic multiplicity 5 and geometric multiplicity 3, and $\lambda = 4$, with algebraic multiplicity 3 and geometric multiplicity 1.

    The characteristic polynomial will have degree $5 + 3 = 8$, so the complex vector space must be eight dimensional. We need to find the right basis, under which the linear operator's representation is in Jordan form. Even better, we can just choose a matrix that is already in Jordan form, and have our operator be multiplication by this matrix. We need exactly one 3×3 Jordan block with $\lambda = 4$, and three Jordan blocks with $\lambda = 3$, with a total size of 5. This can be either two 2×2 blocks and one 1×1 block, or one 3×3 block and two 1×1 block. Then, we assemble these four blocks in any order to form a Jordan matrix.
    Let $A = \begin{pmatrix} 4&1&0&0&0&0&0&0\\ 0&4&1&0&0&0&0&0\\ 0&0&4&0&0&0&0&0\\ 0&0&0&3&1&0&0&0\\ 0&0&0&0&3&0&0&0\\ 0&0&0&0&0&3&1&0\\ 0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&3 \end{pmatrix}$ or $A = \begin{pmatrix} 4&1&0&0&0&0&0&0\\ 0&4&1&0&0&0&0&0\\ 0&0&4&0&0&0&0&0\\ 0&0&0&3&1&0&0&0\\ 0&0&0&0&3&1&0&0\\ 0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&3 \end{pmatrix}$. Then $L : \mathbb{C}^8 \to \mathbb{C}^8$ (or $L : \mathbb{R}^8 \to \mathbb{R}^8$) via $L(x) = Ax$ satisfies the conditions of the problem.
    Note: The vector space, if real, can be greater than eight dimensional, if the characteristic polynomial contains quadratic irreducible factors. However, in this case there is no Jordan canonical form, and the problem becomes much harder to solve in general.

15. Find all 2×2 complex matrices that are simultaneously anti-symmetric and unitary.

    Let $A = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$, an anti-symmetric matrix with complex entries. (note: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = -\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ yields $a = d = 0, b = -c$). Since A is unitary, its eigenvalues must each be of unit length, hence $|A| = a^2$ must be of unit length, so $|a| = 1$. We write $a = e^{i\theta}$; so $A = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{i\theta} & 0 \end{pmatrix}$. For any θ, this satisfies the requirements: it is anti-symmetric by inspection, and unitary since the columns are orthonormal.

16. (extra credit) Prove that every probability matrix has eigenvalue $\lambda = 1$.

    Let A be a probability matrix; that is, its entries are nonnegative real numbers, and the column sums are all 1. Set $x = (1, 1, \ldots, 1)$, a row vector. We have $xA = x$; taking transposes we have $(xA)^T = A^T x^T = x^T$. Hence x^T is an eigenvector of A^T, with eigenvalue 1. However, the matrices A and A^T have the same characteristic polynomials (this is true for any matrix because $|\lambda I - A| = |(\lambda I - A)^T| = |(\lambda I)^T - A^T| = |\lambda I - A^T|$), hence 1 is also an eigenvalue of A.