Math 522 Exam 9 Solutions

1. Calculate $(1 \star \sigma \star E)(6)$.
   Bonus: Calculate $(1 \star 1 \star 1 \star 1 \star 1 \star 1)(6)$.

   For convenience, we first calculate $d(1) = 1, d(2) = d(3) = 2, d(6) = 4, \sigma(1) = 1, \sigma(2) = 3, \sigma(3) = 4, \sigma(6) = 12$.

   Direct method: $(1 \star \sigma)(1)E(6) + (1 \star \sigma)(2)E(3) + (1 \star \sigma)(3)E(2) + (1 \star \sigma)(6)E(1)$. $(1 \star \sigma)(1) = 1(1)\sigma(1) = 1$*;* $(1 \star \sigma)(2) = 1(1)\sigma(2) + 1(2)\sigma(1) = 3 + 1 = 4$*;* $(1 \star \sigma)(3) = 1(1)\sigma(3)+1(3)\sigma(1) = 4+1 = 5$*;* $(1\star\sigma)(6) = 1(1)\sigma(6)+1(2)\sigma(3)+1(3)\sigma(2)+1(6)\sigma(1) = 12 + 4 + 3 + 1 = 20$. *Putting it all together, we get* $1 \cdot 6 + 4 \cdot 3 + 5 \cdot 2 + 20 \cdot 1 = 48$.

   Alternate method 1: Since $\sigma = 1 \star E$*, we rewrite* $1 \star \sigma \star E = 1 \star 1 \star E \star E$*. We then use* $d = 1 \star 1, nd = E \star E$*, to get* $d \star nd$*. Hence, we calculate* $d(1)6d(6) + d(2)3d(3) + d(3)2d(2) + d(6)1d(1) = 1 \cdot 6 \cdot 4 + 2 \cdot 3 \cdot 2 + 2 \cdot 2 \cdot 2 + 4 \cdot 1 \cdot 1 = 48$.

   Alternate method 2: We again start with $1\star\sigma\star E = 1\star1\star E\star E$*, but this time use* $\sigma = 1\star E$ *twice, to get* $\sigma \star \sigma$*. Hence, we calculate* $\sigma(1)\sigma(6) + \sigma(2)\sigma(3) + \sigma(3)\sigma(2) + \sigma(6)\sigma(1) = 1 \cdot 12 + 3 \cdot 4 + 4 \cdot 3 + 12 \cdot 1 = 48$.

   BONUS: Because $1 \star 1 = d$*, we get* $(1 \star 1 \star 1 \star 1 \star 1 \star 1)(6) = (d \star d \star d)(6) = (d \star d)(1)d(6) + (d \star d)(2)d(3) + (d \star d)(3)d(2) + (d \star d)(6)d(1)$*. We now need* $(d \star d)(1) = d(1)d(1) = 1, (d \star d)(2) = d(1)d(2) + d(2)d(1) = 4, (d \star d)(3) = d(1)d(3) + d(3)d(1) = 4, (d\star d)(6) = d(1)d(6) + d(2)d(3) + d(3)d(2) + d(6)d(1) = 1 \cdot 4 + 2 \cdot 2 + 2 \cdot 2 + 4 \cdot 1 = 16$. *Hence the answer is* $1 \cdot 4 + 4 \cdot 2 + 4 \cdot 2 + 16 \cdot 1 = 36$.

2. Prove that $x^{13} \equiv x \pmod{78}$, for every integer x.

   First, we note that $x^{13} \equiv x \pmod{13}$ *holds for all* x*, by Fermat's little theorem. Next, we will show that* $x^{13} \equiv x \pmod{2}$ *holds for all* x*; it suffices to show this for* $x = 0, 1$*, since* $\{0, 1\}$ *is a CRS. We immediately see that* $0^{13} \equiv 0, 1^{13} \equiv 1 \pmod{2}$. *Finally, we will show that* $x^{13} \equiv x \pmod{3}$ *holds for all* x*; it suffices to show this for* $x = 0, 1, -1$*, since* $\{0, 1, -1\}$ *is a CRS. We immediately see that* $0^{13} \equiv 0, 1^{13} \equiv 1, (-1)^{13} \equiv -1 \pmod{3}$. *But now since* $2, 3, 13$ *are pairwise relatively prime, we have* $x^{13} \equiv x \pmod{13 \cdot 2 \cdot 3}$*, which solves the problem since* $13 \cdot 2 \cdot 3 = 78$.

3. Exam grades: 105, 104, 102, 102, 101, 97, 83, 80, 77, 75, 72, 71, 61, 59