

Math 522 Exam 6 Solutions

1. Find a reduced residue system modulo 15 consisting entirely of primes.

The 'standard' reduced residue system is $\{1, 2, 4, 7, 8, 11, 13, 14\}$. Many of these are primes already, but we need to replace $1, 4, 8, 14$, by finding primes congruent to them modulo 15. $1 + 15 = 16$, but $1 + 30 = 31$. $4 + 15 = 19, 8 + 15 = 23, 14 + 15 = 29$. Hence $\{31, 2, 19, 7, 23, 11, 13, 29\}$ is a solution.

BONUS: $1 \equiv 1 \cdot 1 \equiv 2 \cdot 8 \equiv 4 \cdot 4 \equiv 7 \cdot 13 \equiv 11 \cdot 11 \equiv 14 \cdot 14$.

So $31^{-1} = 31, 2^{-1} = 23, 19^{-1} = 19, 7^{-1} = 13, 23^{-1} = 2, 11^{-1} = 11, 13^{-1} = 7, 29^{-1} = 29$.

An important theorem of Dirichlet's on primes in arithmetic progressions states that you may replace any element from the reduced residue system with *INFINITELY* many different primes.

2. Fix $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Suppose that $R = \{r_i : 1 \leq i \leq m\}$ is a complete residue system modulo m , and that $S = \{s_j : 1 \leq j \leq n\}$ is a complete residue system modulo n . Prove that $T = nR + mS = \{nr_i + ms_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a complete residue system modulo mn .

The strategy will be to use the main theorem proved in class for this section. To prove that T is a complete residue system, we need to show that any two different elements of T are incongruent, and we need to show that $|T| = mn$.

Suppose two elements of T were congruent; then $nr_i + ms_j \equiv nr'_i + ms'_j \pmod{mn}$. Because $\gcd(m, n) = 1$, this is logically equivalent (by an earlier theorem) to $nr_i + ms_j \equiv nr'_i + ms'_j \pmod{m}$ AND $nr_i + ms_j \equiv nr'_i + ms'_j \pmod{n}$.

From the first equivalence, we may conclude that $nr_i \equiv nr'_i \pmod{m}$. But, since $\gcd(n, m) = 1$, we may divide by n to get $r_i \equiv r'_i \pmod{m}$. Since r_i, r'_i are in R , they must in fact be the same element of R .

From the second equivalence, we proceed similarly. We conclude that $ms_j \equiv ms'_j \pmod{n}$. But, since $\gcd(n, m) = 1$, we may divide by m to get $s_j \equiv s'_j \pmod{n}$. Since s_j, s'_j are in S , they must in fact be the same element of S .

But then $nr_i + ms_j$ and $nr'_i + ms'_j$ are in fact the same element of T . Hence, any two different elements of T are incongruent.

We now show that $|T| = mn$. If $nr_i + ms_j = nr'_i + ms'_j$, then in particular $nr_i + ms_j \equiv nr'_i + ms'_j \pmod{mn}$. But we've already shown this to be impossible in the first part of the problem, hence the elements of T are unequal (i.e. all different). Note: This proof is nontrivial; if $\gcd(m, n) \neq 1$, then $|T|$ might not be mn ; for example if $R = \{0, 1\}, S = \{0, 1\}, m = n = 2$, we have $T = \{0, 2, 4\}$.

3. Exam grades: 95, 95, 94, 92, 91, 88, 83, 82, 80, 78, 76, 66, 64, 57