Math 522 Exam 1 Solutions

1. Please write one hundred in eight ways: in base 8,9,10,11,12,13,14,15. If needed, use 'A' to represent the digit ten, 'B' for eleven, and so on.
   BONUS: write one hundred in factoradic.

   8: $1 \cdot 64 + 4 \cdot 8 + 4 = (144)_8$
   9: $1 \cdot 81 + 2 \cdot 9 + 1 = (121)_9$
   10: $1 \cdot 100 + 0 \cdot 10 + 0 = (100)_{10}$
   11: $9 \cdot 11 + 1 = (91)_{11}$
   12: $8 \cdot 12 + 4 = (84)_{12}$
   13: $7 \cdot 13 + 9 = (79)_{13}$
   14: $7 \cdot 14 + 2 = (72)_{14}$
   15: $6 \cdot 15 + 10 = (6A)_{15}$
   Factoradic: $4 \cdot 24 + 0 \cdot 6 + 2 \cdot 2 + 0 \cdot 1 = (4020)_!$

2. A nonempty set of integers J that fulfills the following two conditions is called an integral ideal:

   (a) If n, m are in J, then $n + m$ and $n - m$ are in J; and

   (b) If n is in J and r is any integer, then rn is in J.

   Further, for any integer m let $J_m = \{mk : k \in \mathbb{Z}\}$, the set of all integer multiples of m. You may assume that J_m is an integral ideal. Prove that every integral ideal J is, in fact, equal to J_m for some $m \in \mathbb{Z}$.

   If J contains no positive elements, then it contains no negative elements either (if $x \in J$, with x negative, then $(-1)x \in J$ by (b), which would be a forbidden positive element) and therefore $J = \{0\}$ since J is nonempty. In this case, $J = J_0$.

   Otherwise, J has at least one positive element, and therefore by the well-ordering of $\mathbb{N}$ must have a minimal positive element, which we will call m. By (b), all integer multiples of m are in J, and hence $J_m \subseteq J$. It remains to show that $J \subseteq J_m$.

   Suppose by way of contradiction that there is some $n \in J$ but $n \notin J_m$. Using the division algorithm, we divide n by m to get $n = qm + r$. We know that $n \in J$ (hypothesis), and $qm \in J$ (because $qm \in J_m$, and $J_m \subseteq J$), so by (a) we must have $r = n - qm \in J$. But the division algorithm guarantees that $0 \leq r < m$, which contradicts the fact that m is minimal and positive in J. Hence any element of J must also be in J_m, which proves $J \subseteq J_m$ and hence $J = J_m$.

   If we replace $\mathbb{Z}$ with another ring, we can still consider subsets J with the two properties above; they are called ideals and are very important in algebra. Ideals like J_m (generated by one element) are called principal ideals. In the integers, every ideal is principal; rings where this is true are called 'principal ideal domains', or PID's for short. $\mathbb{R}[x]$, the set of all polynomials with real coefficients, is a PID, but $\mathbb{Z}[x]$ and $\mathbb{R}[x, y]$ are not.

3. Exam grades: 101, 97, 90, 88, 80, 80, 80, 80, 79, 78, 78, 75, 75, 73, 69