

## Math 254 Exam 9 Solutions

1. Carefully define the term “basis”. Give two examples in  $\mathbb{R}^2$ .

A basis is a set of vectors that is independent and maximal. Four examples are  $\{(1, 0), (0, 1)\}$ ,  $\{(1, 1), (3, 2)\}$ ,  $\{(1, 0), (3, 7)\}$ ,  $\{(-8, 8), (-8, 7)\}$ .

2. Consider the basis  $S = \{u_1, u_2\} = \{(1, -2), (2, -5)\}$  of  $\mathbb{R}^2$ , and the linear operator  $F(x, y) = (2x + 3y, 4x - 5y)$ . Find the matrix representation  $[F]_S$ .

The change-of-basis matrix from the standard basis  $E$  to  $S$  is  $P = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$ . We also have  $[F]_E = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$ . We now calculate  $P^{-1} = \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix}$ , and now  $[F]_S = P^{-1}[F]_E P = \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$ .

3. Let  $V$  be the vector space of functions that have as a basis  $S = \{u_1, u_2, u_3, u_4, u_5\} = \{1, \sin \theta, \cos \theta, \sin 2\theta, \cos 2\theta\}$ . Let  $D$  be the differential operator on  $V$ . Find the matrix representation  $[D]_S$ .

BONUS: What is the nullity and rank of  $D$ ?

We calculate  $D(u_1) = 0, D(u_2) = u_3, D(u_3) = -u_2, D(u_4) = 2u_5, D(u_5) = -2u_4$ .

Therefore  $[D]_S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$ . BONUS:  $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  is  $[D]_S$  in row

canonical form. This is rank four;  $D$  must have the same rank. Since  $S$  has dimension 5, the nullity of  $D$  is  $5 - 4 = 1$ .

4. Set  $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ . Find two other matrices similar to  $A$ .

Many solutions exist; choose any invertible matrix  $P$ , calculate  $P^{-1}$ , and  $P^{-1}AP$  will be similar to  $A$ . For  $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ , we have  $P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ , and  $P^{-1}AP = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \sim A$ . For  $P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $P^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ , and  $P^{-1}AP = \begin{bmatrix} -2 & -4 \\ 3 & 5 \end{bmatrix} \sim A$ .

5. Prove that, for any square matrices  $A, B$ , if  $A$  is similar to  $B$ , then  $B$  must be similar to  $A$ .

1. Suppose that  $A$  is similar to  $B$ .

2. Hence, there is some invertible matrix,  $P$ , such that  $A = P^{-1}BP$ .

3. Multiply on the left by  $P$ , and on the right by  $P^{-1}$  to get  $PAP^{-1} = PP^{-1}BPP^{-1} = IBP^{-1} = B$ .

4. Set  $Q = P^{-1}$ .  $Q$  is invertible, and  $Q^{-1} = (P^{-1})^{-1} = P$ .

5. There is an invertible matrix, namely  $Q$ , such that  $B = Q^{-1}AQ$ .

6. Hence,  $B$  is similar to  $A$ .