Math 254 Exam 7a Solutions

1. Carefully define the term "linear combination".

   A linear combination of some objects (vectors, variables, etc.) is the sum of those objects, each multiplied by a scalar.

2. Choose ALWAYS or SOMETIMES or NEVER.

   (a) An inner product space is ALWAYS a vector space.
   (b) A normed space is SOMETIMES an inner product space.
   (c) A vector space is SOMETIMES an inner product space.
   (d) An inner product space is ALWAYS a normed space.
   (e) A normed space is ALWAYS a vector space.
   (f) A vector space is SOMETIMES a normed space.

3. Carefully state the three axioms of an inner product.

   (I1) $<a\overline{u_1}+b\overline{u_2},\overline{v}>= a<\overline{u_1},\overline{v}> +b<\overline{u_2},\overline{v}>$
   (I2) $<\overline{u},\overline{v}>=<\overline{v},\overline{u}>$
   (I3) $<\overline{u},\overline{u}>\ \geq 0$; further, $<\overline{u},\overline{u}>= 0$ precisely when $\overline{u}=\overline{0}$.

   For the next two questions, consider the vector space $P(t)$ with inner product given by $<u,v>=\int_0^1 u(t)v(t)dt$. Let $f(t)=t$, and $g(t)=at+1$, for some unknown constant a.

4. For which value(s) of a are f and g orthogonal?

   $<f,g>=\int_0^1 at^2+tdt = at^3/3+t^2/2\Big|_0^1 = a/3+1/2$. f,g are orthogonal precisely when $<f,g>=0$; this occurs for $a=-3/2$.

5. We want to find which value(s) of a cause f,g to have a 60° angle between them. Set up (but do not solve) an equation in a that would answer this question.
   BONUS: Solve the equation.

   $\cos(60°) = 1/2 = \frac{<f,g>}{||f||\ ||g||}$. $<f,g>= a/3+1/2$, as calculated before. $||f|| = \sqrt{\int_0^1 t^2dt} = \sqrt{t^3/3\Big|_0^1} = \sqrt{1/3}$. $||g|| = \sqrt{\int_0^1 (at+1)^2dt} = \sqrt{\int_0^1 a^2t^2+2at+1dt} = \sqrt{a^2t^3/3+at^2+t\Big|_0^1} = \sqrt{a^2/3+a+1}$ Hence, we need to solve $1/2 = \frac{a/3+1/2}{\sqrt{1/3}\sqrt{a^2/3+a+1}}$.

   BONUS: $1/2 = \frac{a/3+1/2}{\sqrt{1/3}\sqrt{1/3}\sqrt{a^2+3a+3}} = \frac{a+3/2}{\sqrt{a^2+3a+3}}$. We cross-multiply to get $a+3/2 = (1/2)\sqrt{a^2+3a+3}$. We now square both sides to get $(1/4)(a^2+3a+3) = (a+3/2)^2 = a^2+3a+9/4$. We rearrange to get $(3/4)a^2+(9/4)a+(6/4) = 0$. We simplify by multiplying both sides by $(4/3)$ to get $a^2+3a+2=0$. We factor this to get $(a+2)(a+1)=0$ and thus $a=-1,-2$. However, $a=-2$ is extraneous; $(-2)/3+1/2 < 0$, so for this value of a, the vectors do not have angle 60° (in fact, they have angle 120°). This leaves only $a=-1$.