Math 254 Exam 6 Solutions

1. Carefully define the Linear Algebra term "independent".

   A set of vectors is independent if no nondegenerate linear combination yields $\bar{0}$.

2. In the vector space $M_{2,3}$ of 2×3 matrices, set:

   $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 7 \\ 10 & 1 & 13 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 5 \\ 8 & 2 & 11 \end{bmatrix}$

   Determine whether or not $\{A, B, C\}$ is independent.

   Let E be the standard basis for $M_{2,3}$. Then $[A]_E = [1\ 2\ 3\ 4\ 0\ 5], [B]_E = [2\ 4\ 7\ 10\ 1\ 13], [C]_E = [1\ 2\ 5\ 8\ 2\ 11]$. We put these row matrices into a larger matrix, which we then put into echelon form:

   $$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 2 & 4 & 7 & 10 & 1 & 13 \\ 1 & 2 & 5 & 8 & 2 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   This is seen to have rank 2, hence $\{A, B, C\}$ is dependent.

   ALTERNATE SOLUTION:

   $$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 5 \\ 4 & 10 & 8 \\ 0 & 1 & 2 \\ 5 & 13 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   This has rank 2, hence $\{A, B, C\}$ is dependent.

3. In the vector space $P_3(x)$ of polynomials of degree at most 3, set $u_1 = x^3 + 3x^2 - 2x + 4, u_2 = 2x^3 + 7x^2 - 2x + 5, u_3 = x^3 + 5x^2 + 2x - 2, u_4 = 2x^3 + 6x^2 - 4x + 5$

   Set $S = span\{u_1, u_2, u_3, u_4\}$. Find the dimension of S, and a basis.

   Let $E = \{x^3, x^2, x, 1\}$ be the usual basis for $P_3(x)$. We have $[u_1]_E = [1\ 3\ -2\ 4], [u_2]_E = [2\ 7\ -2\ 5], [u_3]_E = [1\ 5\ 2\ -2], [u_4]_E = [2\ 6\ -4\ 5]$. We put these row matrices into a larger matrix, which we then put into echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 4 \\ 2 & 7 & -2 & 5 \\ 1 & 5 & 2 & -2 \\ 2 & 6 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has rank 3; hence $\dim S = 3$. A basis for S is $\{x^3 + 3x^2 - 2x + 4, x^2 + 2x - 3, 1\}$.

ALTERNATE SOLUTION:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 7 & 5 & 6 \\ -2 & -2 & 2 & -4 \\ 4 & 5 & -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has rank 3; hence $\dim S = 3$. Because the pivots are in the first, second, and fourth columns, a basis for S is $\{u_1, u_2, u_4\}$.

4. In the vector space $\mathbb{R}^2$, set $S = \{(1,3), (1,4)\}$, a basis. Find the change-of basis matrix from S to the standard basis, and use this matrix to find $[(5,-3)]_S$.

   $P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ consists of S in column form; $Q = P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ is the desired matrix. We find $[(5,-3)]_S = Q \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}$.

5. In the vector space $\mathbb{R}^3$, set $T = \{(1,1,1), (0,1,2), (1,1,3)\}$, a basis. Find $[(1,2,2)]_T$.

   $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ consists of S in column form; $Q = P^{-1} = \begin{bmatrix} 1/2 & 1 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1 & 1/2 \end{bmatrix}$ is the change-of-basis matrix (found by applying ERO's to $[P|I]$ until we achieve $[I|Q]$. We find $[(1,2,2)]_T = Q \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -1/2 \end{bmatrix}$.