

## Math 254 Exam 5 Solutions

1. Carefully define the Linear Algebra term “independent”.

A set of vectors is independent if there is no nondegenerate linear combination of them that yields the zero vector.

The next three questions concern the matrix  $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$ .

All three questions benefit from reducing  $A$  to row echelon form. We do  $-2R_1 + R_2 \rightarrow R_2$ ,  $-3R_1 + R_3 \rightarrow R_3$ ,  $-2R_2 + R_3 \rightarrow R_3$ . This is actually Example 5.5 in the text, on p.189 (except for a typo; the last row of the intermediate step should be 0 4 -6 -2).

The row echelon form is  $B = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ .

2. Set  $S = \text{rowspace}(A)$ . Find a basis for  $S$ . Determine the dimension of  $S$ .

Because  $B$  has two nonzero rows, the rowspace has dimension 2. A basis is  $\{(1, 2, 0, -1), (0, 2, -3, -1)\}$ .

3. Set  $T = \text{columnspace}(A)$ . Find a basis for  $T$ . Determine the dimension of  $T$ .

Because  $B$  has two pivots, the columnspace has dimension 2 (also because of the answer to the previous question). Because the pivots are in the first two columns, a basis for the columnspace is  $\{(1, 2, 3), (2, 6, 10)\}$ .

4. Write the matrix equation  $AX = 0$  as a homogeneous system of equations; find a basis for its solution space.

$$\begin{aligned} x + 2y + 0z - w &= 0 \\ 2x + 6y - 3z - 3w &= 0 \\ 3x + 10y - 6z - 5w &= 0 \end{aligned}$$

We look at  $B$ ;  $z, w$  are free variables, hence the solution space is 2 dimensional. To find a basis, try  $z = 1, w = 0 : (-3, 3/2, 1, 0)$  and  $z = 0, w = 1 : (0, 1/2, 0, 1)$ ; hence  $\{(-3, 3/2, 1, 0), (0, 1/2, 0, 1)\}$  is a basis for the solution space.

5. In the vector space  $\mathbb{R}^3$ , set  $U = \text{span}\{(1, 2, 3), (5, -2, 3)\}$ ,  $V = \text{span}\{(-1, 2, 1)\}$ . Determine the dimensions of each of  $U, V, U \cap V, U + V$ . Justify your answers.

$\dim U = 2$  since the two basis vectors are not scalar multiples of each other.  $\dim V = 1$ .

To find  $\dim(U + V)$ , we put  $\begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 3 \\ -1 & 2 & 1 \end{bmatrix}$  into row echelon form:  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -12 & -12 \\ 0 & 0 & 0 \end{bmatrix}$ .

Hence  $\dim(U + V) = 2$ , and therefore the line  $V$  actually lies in the plane  $U$ . This means that  $V = U \cap V$ , so  $\dim(U \cap V) = 1$ . Alternatively, by Theorem 5.9,  $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$ . We have  $2 = 2 + 1 - \dim(U \cap V)$ ; we solve this to get  $\dim(U \cap V) = 1$ .