

## Math 254 Exam 2b Solutions

1. Carefully define the term “dimension” as it applies to vector spaces. Give two examples: a three-dimensional vector space, and an infinite-dimensional vector space.

The dimension of a vector space is the number of elements in a basis.  $\mathbb{R}^3$ , the set of all 3-vectors, is a three-dimensional vector space.  $\mathbb{R}[x]$ , the set of all polynomials with real coefficients, is an infinite-dimensional vector space.

2. Write the system as a matrix equation.

$$\begin{bmatrix} 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 9 & -2 & 5 \\ 2 & 6 & 3 & -1 \\ 5 & 15 & 0 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -3 \\ 11 \\ 17 \end{bmatrix}$$

3. Write the system as an augmented matrix; put this matrix in echelon form. Justify each step using elementary row operations. Using the echelon form, find all solutions to the system (if any).

$$\begin{bmatrix} 1 & 3 & -3 & 0 & | & -4 \\ 0 & 0 & 1 & 3 & | & -1 \\ 3 & 9 & -2 & 5 & | & -3 \\ 2 & 6 & 3 & -1 & | & 11 \\ 5 & 15 & 0 & -7 & | & 17 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -3 & 0 & | & -4 \\ 0 & 0 & 1 & 3 & | & -1 \\ 0 & 0 & 7 & 5 & | & 9 \\ 0 & 0 & 9 & -1 & | & 19 \\ 0 & 0 & 15 & -7 & | & 37 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -3 & 0 & | & -4 \\ 0 & 0 & 1 & 3 & | & -1 \\ 0 & 0 & 0 & -16 & | & 16 \\ 0 & 0 & 0 & -28 & | & 28 \\ 0 & 0 & 0 & -52 & | & 52 \end{bmatrix} \rightarrow$$
$$\rightarrow \begin{bmatrix} 1 & 3 & -3 & 0 & | & -4 \\ 0 & 0 & 1 & 3 & | & -1 \\ 0 & 0 & 0 & -16 & | & 16 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first step combines  $-3R_1 + R_3 \rightarrow R_3$ ,  $-2R_1 + R_4 \rightarrow R_4$ ,  $-5R_1 + R_5$ . The second step combines  $-7R_2 + R_3 \rightarrow R_3$ ,  $-9R_2 + R_4 \rightarrow R_4$ ,  $-15R_2 + R_5 \rightarrow R_5$ . The third step combines  $(-28/16)R_3 + R_4 \rightarrow R_4$ ,  $(-52/16)R_3 + R_5 \rightarrow R_5$ . The end result is in echelon form;  $x, z, w$  are basic, and  $y$  is free; set  $y = a$ . We solve  $-16w = 16$  to get  $w = -1$ . We solve  $z + 3(-1) = -1$  to get  $z = 2$ . We solve  $x + 3y - 3z = -4$  to get  $x = 2 - 3a$ . Putting it all together, the solution set is  $\{(2 - 3a, a, 2, -1)\}$ , for any real number  $a$ .

4. Write the system as an augmented matrix; put this matrix in row canonical form. Justify each step using elementary row operations. Using the row canonical form, find all solutions to the system (if any).

$$\begin{aligned}
& \left[ \begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & -16 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 3 & -3 & 0 & -4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \\
& \rightarrow \left[ \begin{array}{cccc|c} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

We begin where the previous problem left off. The first step is  $(-1/16)R_3 \rightarrow R_3$ . The second step is  $-3R_3 + R_2 \rightarrow R_2$ . The third step is  $3R_2 + R_1 \rightarrow R_1$ . We see immediately that  $w = -1, z = -2, x = 2 - 3y; y = a$ , a free parameter. This yields the same solution as above.

5. Write the system as an augmented matrix; put this matrix in echelon form using partial pivoting. Justify each step using elementary row operations.

$$\begin{aligned}
& \left[ \begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 1 & 3 & -1 \\ 3 & 9 & -2 & 5 & -3 \\ 2 & 6 & 3 & -1 & 11 \\ 1 & 3 & -3 & 0 & -4 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & 46/5 & -66/5 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & -3 & 7/5 & -37/5 \end{array} \right] \rightarrow \\
& \rightarrow \left[ \begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & 46/5 & -66/5 \\ 0 & 0 & -3 & 7/5 & -37/5 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 0 & 12/5 & -12/5 \\ 0 & 0 & 0 & 52/5 & -52/5 \\ 0 & 0 & 0 & 16/5 & -16/5 \end{array} \right] \rightarrow \\
& \rightarrow \left[ \begin{array}{cccc|c} 5 & 15 & 0 & -7 & 17 \\ 0 & 0 & 3 & 9/5 & 21/5 \\ 0 & 0 & 0 & 12/5 & -12/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

Before the first step, we performed  $R_1 \leftrightarrow R_5$ , to put the 5 in the upper left corner. The next step combines  $(-3/5)R_1 + R_3 \rightarrow R_3, (-2/5)R_1 + R_4 \rightarrow R_4, (-1/5)R_1 + R_5 \rightarrow R_5$ . The next step was  $R_2 \leftrightarrow R_4$ , to put the 3 in the upper left corner of the remaining box (it would have also been correct to do  $R_2 \leftrightarrow R_5$ ). The next step combines  $(-1/3)R_2 + R_3 \rightarrow R_3, (2/3)R_2 + R_4 \rightarrow R_4, R_2 + R_5 \rightarrow R_5$ . The final step combines  $(-52/12)R_3 + R_4 \rightarrow R_4, (-16/12)R_3 + R_5 \rightarrow R_5$ .