Math 151 Final Review: Fall 2007 Vadim Ponomarenko
All questions taken from http://www.math.sdsu.edu/math_final_exam.htm

Essential Concepts:

Integration by Parts:

$\int x\cos(5x)\,dx$ $\quad \int \frac{\ln(x)}{\sqrt{x}}dx$ $\quad \int xe^{-x/2}dx$ $\quad \int xe^{-4x}dx$ $\quad \int x^2\ln(x)\,dx$

$\int x\sin(x/3)\,dx$ $\quad \int \frac{\ln(x)}{x^{1/3}}dx$ $\quad \int xe^{-x/2}dx$ $\quad \int \arcsin(x)\,dx$ $\quad \int xe^{-x/2}dx$

$\int x\cosh\left(\frac{x}{4}\right)dx$ $\quad \int x\cos\left(\frac{x}{2}\right)dx$ $\quad \int x\cosh(5x)\,dx$

(yes, the same integral has appeared 3 times)

4 (10 pts.) Determine whether the improper integral

$$\int_{-\infty}^{0} xe^{x}\,dx$$

converges or diverges, and its value in case it converges.

Partial Fractions:

$\int \frac{x-19}{(x-3)(x+1)}dx$ $\quad \int \frac{4x^2+7x-3}{(x+1)^2(x-2)}dx$ $\quad \int \frac{x-23}{x^2+3x-10}dx$ $\quad \int \frac{7x+11}{(x+3)(x-2)}dx$

$\int \frac{x+23}{x^2-3x-10}dx$ $\quad \int \frac{x-11}{x^2+3x-4}dx$ $\quad \int \frac{4x^2+2x+19}{(x-2)(x^2+9)}dx$ $\quad \int \frac{3x+14}{x^2+x-6}dx$ $\quad \int \frac{x-1}{x^2-7x+12}dx$

4. (5 pts) Find the *form* of the partial fraction decomposition for the expression

$$\frac{3x^4+13x^3+7x^2-17x+14}{(x+2)(x^2+1)(x+3)^2}$$

You do not need to solve for the coefficients.

$$\int \frac{x+26}{x^2-2x-8}dx$$

Improper Integrals:

4 (10 pts.) Determine whether the improper integral

$$\int_{-\infty}^{0} xe^{x}\,dx$$

converges or diverges, and its value in case it converges.

$$\int_0^\infty 4^{-x}\,dx$$

4 (10 pts.) Determine whether the improper integral $\int_{2\sqrt{3}}^{\infty} \frac{1}{x^2+4}dx$ converges or diverges, and its value in the case of convergence.

(b) (5 pts) Determine whether the improper integral

$$\int_0^1 \frac{\ln(x)}{x^{1/3}}dx$$

converges or diverges and its value in the case of convergence.

5. (10 pts.) Use the integral test to determine whether the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln^4(n)}$$

converges or diverges.

5 (10 pts.) Use the integral test to determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+4}$$

converges or diverges (you need not justify the applicability of the test).

6. (10 pts.) Use the integral test to determine if the infinite series

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

converges or diverges. (You need not justify the applicability of the test.)

5. (5 pts.) Determine whether the improper integral

$$\int_2^4 \frac{1}{x-2} dx$$

converges or diverges, and the value of the improper integral in case of convergence.

4 (5 pts.) Determine whether the improper integral

$$\int_{\sqrt{\ln(2)}}^{\infty} xe^{-x^2} dx$$

converge or diverges, and its value in case it converges (express your response in a simplified form).

4 (5 pts.) Determine whether the improper integral

$$\int_0^{\infty} \frac{x}{(x^2+1)^2} dx$$

converge or diverges, and its value in case it converges.

4 (5 pts.) Determine whether the improper integral

$$\int_0^{\infty} \frac{1}{16+4x^2} dx$$

converge or diverges, and its value in case it converges.

(Absolute) Ratio Test:

6 (7 pts.) Use the ratio test to determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{10^n}$$

converges absolutely or diverges.

6 (10 pts.) Determine whether

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$$

converges absolutely or conditionally.

10. (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!}$$

converges absolutely, converges conditionally or diverges.

10 (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} \frac{n!}{10^n}$$

converges or diverges.

8 (8 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{e^n}$$

converges absolutely, converges conditionally or diverges. Justify your response.

10 (5 pts.) Determine whether the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{2^n}$$

converges or diverges.

4 (10 pts.) Use the ratio test to determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n!}$$

converges absolutely or diverges.

Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$ converges or diverges.

10 (5 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n!}$$

converges absolutely, converges conditionally or diverges.

Interval of Convergence (absolute ratio test):

9 (8 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{4^n}(x-2)^n$$

(You need not worry about the endpoints of the interval).

7 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x+4)^n}{n^2}$. (You need not investigate the series at the endpoints of the interval.)

7 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{n}{3^n}(x-4)^n$$

(you need not investigate the series at the endpoints of the interval)

8. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n n^{3/2}}.$$

(You need not investigate the series at the endpoints of the interval.)

6 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{2^n n^2}.$$

(You need not investigate the series at the endpoints of the interval.)

12. (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{4^n n^{1/3}}.$$

(You need not investigate the series at the endpoints of the interval.)

12 (5 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{n^2}(x-4)^n.$$

(You need not investigate the series at the endpoints of the interval.)

12 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{n^3}{3^n}(x-4)^n.$$

(You need not investigate the series at the endpoints of the interval.)

12 (10 pts.) Determine the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2^n}(x+6)^n.$$

(You need not investigate the series at the endpoints of the interval.)

Taylor Series:

10. We have

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

($\arctan(x) = \tan^{-1}(x)$).

a) (5 pts.) Determine

$$\lim_{x\to 0} \frac{\arctan(x^2) - x^2}{x^6}$$

by using the Maclaurin series of $\arctan(x)$ (**Do not use L'Hôpital's rule**)

b) (5 pts.) Let

$$F(x) = \int_0^x \frac{\arctan(t^2) - t^2}{t^6} dt.$$

Determine the first three terms of the Maclaurin series for F.

8 (10 pts.) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, determine the Maclaurin series for $F(x) = \int_0^x \frac{e^t - 1 - t}{t^2} dt$. (Display the first 4 terms and the term involving x^n. You may leave your answer in terms of the factorial.)

$$F(x) = \int_0^x \frac{1}{1+t^4} dt.$$

Determine the Maclaurin series for F up to the term that has x^{13}

Hint: The starting point can be the geometric series.

9. (10 pts.) Given that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, -1 \le x \le 1$$

determine the first four nonzero terms of the Maclaurin series for

$$\frac{d}{dx}\arctan(x^2)$$

Simplify as much as possible.

Determine the Maclaurin polynomial of order 3 for $(x+1)^{1/3}$.

8 (10 pts.) Let

$$F(x) = \int_0^x \frac{1}{1-t^2} dt.$$

Determine the Maclaurin series for F (display the first 3 terms and the term involving x^{2n+1}).

13. (5 pts.) Determine the Maclaurin series of

$$f(x) = \frac{1}{1+x^2}$$

and the open interval of convergence of the series (Display the first 4 terms and the term that involves x^{2n} for an arbitrary positive integer n).

Hint: Think of the geometric series.

14. (5 pts.) Let $f(x) = \sin(x)$. Determine the first 4 terms of the Taylor series of f in powers of $(x - \pi/6)$.

13 (5 pts.) Let

$$F(x) = \int_0^x t^2 e^{-t^2}\,dt.$$

Determine the first 4 nonzero terms of the Maclaurin series for F and the coefficient of x^{2n+3}, where n is an arbitrary nonnegative integer.
Hint: Start with the Maclaurin series for the natural exponential function.

Let

$$f(x) = \begin{cases} \dfrac{\sin(x) - x}{x^3} & \text{if } x \neq 0, \\ -\dfrac{1}{6} & \text{if } x = 0. \end{cases}$$

Determine the first 3 nonzero terms of the Maclaurin series for f.and the coefficient of x^{2n-2}, where n is an arbitrary positive integer.
Hint: Start with the Maclaurin series for sine.

13 (5 pts.) Let $f(x) = \ln(x)$. Determine the part of the Taylor series for f based at 2 up to the term that has $(x-2)^3$.

14. (5 pts.) Let $f(x) = \sin(x)$. Determine the first 4 terms of the Taylor series of f in powers of $(x - \pi/6)$.

13 (5 pts.) Let $f(x) = \sin(x)$. Determine the part of the Taylor series for f based at $\pi/2$ up to the term that has $(x - \pi/6)^3$.

14 (5 pts.) Given that

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$$

Determine the Maclaurin series for $\arcsin(x)$ up to the term that has x^7.

Differential Equations:

7. (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + 2ty(t) = 2te^{-t^2},\ y(2) = 1.$$

7 (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + \frac{y(t)}{t} = \sin(t),\ y(\pi) = 3$$

Assume that $t > 0$.

7 (10 pts.) Make use of the technique of an integrating factor to determine the solution of the initial value problem

$$\frac{dy}{dt} + y(t) = e^{-t},\ y(0) = 2.$$

Determine the solution of the initial value problem

$$\frac{dy}{dt} = -\frac{1}{4}y(t) + 3t,\ y(4) = 2.$$

) Determine the solution of the initial value problem

$$\frac{dy}{dx} = y^2 e^{-x},\ y(0) = 2$$

Determine the solution of the initial value problem

$$\frac{dy}{dt} = \frac{ty^2}{\sqrt{1+t^2}},\ y(0) = 3$$

Determine the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y^2}{1+x^2},\ y(1) = -\frac{2}{\pi}$$

Determine the solution of the initial value problem

$$\frac{dy}{dt} = \sin(t)\, y^2,\ y(\pi) = 4$$

Polar Coordinates:

9 Let

$$r = f(\theta) = 1 + 2\cos(\theta).$$

a) (5 pts) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a maximum or minimum value.

b) (10 pts.) Sketch the graph of $r = f(\theta)$ as a polar equation in the xy-plane (ie, $x = r\cos(\theta)$, $y = r\sin(\theta)$).

11. Let $f(\theta) = 2 + 4\cos(\theta)$.

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f\ f(\theta) = 0$.

b) (8 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian xy-plane if r and θ are polar coordinates, i.e., $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$$r = f(\theta) = 2 - \cos(\theta).$$

a) (5 pts) Sketch the graph of $r = f(\theta)$ the Cartesian θr-plane on the interval $[0, 2\pi]$.

b) (10 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian xy-plane if r and θ are polar coordinates ($x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = 3\sin(2\theta).$$

(a) (5 pts) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a maximum or minimum value.

(b) (5 pts.) Sketch the graph of $r = f(\theta)$ as a polar equation in the xy-plane (ie, $x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = 1 - 2\cos(\theta).$$

a) (5 pts) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a maximum or minimum value.

b) (10 pts.) Sketch the graph of $r = f(\theta)$ as a polar equation in the xy-plane (ie, $x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = 1 - 2\cos(\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \le \theta \le 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = \cos(2\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \le \theta \le 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = 2 - 2\sin(\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local maximum or minimum value.

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \le \theta \le 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

$$r = f(\theta) = \cos(2\theta).$$

a) (5 pts.) Sketch the graph of $r = f(\theta)$ in the Cartesian θr-plane on the interval $[0, 2\pi]$. Indicate the values of θ at which $f(\theta) = 0$ and the points at which f attains a local

b) (5 pts.) Sketch the graph of $r = f(\theta)$, where $0 \le \theta \le 2\pi$, as a polar equation in the xy-plane (i.e., $x = r\cos(\theta)$, $y = r\sin(\theta)$).

Important Concepts:

Trigonometric Integrals:

$$\int \cos^2(x)\sin^3(x)\,dx \qquad \int \sin^2(4x)\,dx \qquad \int \sin^3(x)\cos^2(x)\,dx \qquad \int \sin^2(3x)\,dx. \qquad \int \cos^2\left(\frac{3x}{4}\right)dx.$$

6. (5 pts.) Determine the volume of the solid that is obtained by revolving the graph of

$$f(x) = \cos\left(\frac{x}{2}\right)$$

on the interval $[-\pi, \pi]$ about the x-axis.

Limit Comparison Test:

7 (10 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{n^2}{4n^4 - 10n + 2}$$

converges or diverges.

Hint: Use the limit comparison test.

7. (10 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 + 3}$$

converges absolutely, converges conditionally, or diverges.

6 (10 pts.) Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3 + 1}$ converges absolutely, converges conditionally or diverges.

11.(10 pts.) Determine whether the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1}$$

converges absolutely, converges conditionally or diverges.

11 (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{(n^2 + 1)^{2/3}}$$

converges absolutely, converges conditionally or diverges.

Integral Comparison Test:

4 (5 pts.) Does the improper integral

$$\int_1^\infty \frac{\sin^2(x)}{x^2}dx$$

converge or diverge? Justify your response.
Hint: Make use of a comparison theorem.

5 (5 pts.) Use the comparison test to determine whether the improper integral

$$\int_0^1 \frac{e^x}{x^{3/2}}dx$$

converges or diverges (if you claim that the integral converges, you do not have to evaluate the value of the integral).

5 (5 pts.) Use a comparison test to determine whether the improper integral

$$\int_1^\infty \frac{\sin^2(x)}{e^x}dx$$

converges or diverges.

5 (5 pts.) Use a comparison test to determine whether the improper integral

$$\int_1^\infty \frac{e^{-x^2}}{x^4}dx$$

converges or diverges. You need not determine the value of the improper integral in case of convergence.

Volumes:

6. (5 pts.) Determine the volume of the solid that is obtained by revolving the graph of

$$f(x) = \cos\left(\frac{x}{2}\right)$$

on the interval $[-\pi, \pi]$ about the x-axis.

6 (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of

$$f(x) = \frac{1}{x^2+4}$$

and the interval $[0, 2]$ about the vertical axis.

6 (10 pts.) Determine the volume of the solid that is obtained by revolving the region between the graph of

$$f(x) = \sqrt{\frac{x}{x^2+4}}$$

and the interval $\left[\sqrt{e^2-4}, \sqrt{e^3-4}\right]$ about the x-axis.

6 (10 pts.) Determine the area of the surface that is obtained by revolving the graph of

$$f(x) = x^3$$

and the interval $[0, 2]$ about the x-axis.

<u>Integral test:</u>

11 (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n \ln^2(n)}$$

converges absolutely, converges conditionally or diverges.

11 (10 pts.) Determine whether the infinite series

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$$

converges absolutely, converges conditionally or diverges.

5 (10 pts.) Use the integral test to determine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+4}$$

converges or diverges (you need not justify the applicability of the test).

5. (10 pts.) Use the integral test to determine whether the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^4(n)}$$

converges or diverges.

6. (10 pts.) Use the integral test to determine if the infinite series

$$\sum_{n=1}^{\infty} n e^{-n^2}$$

converges or diverges. (You need not justify the applicability of the test.)