ON THE SET OF CATENARY DEGREES OF FINITELY GENERATED CANCELLATIVE COMMUTATIVE MONOIDS

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ABSTRACT. The catenary degree of an element s of a cancellative commutative monoid S is a nonnegative integer measuring the distance between the irreducible factorizations of s. The catenary degree of the monoid S, defined as the supremum over all catenary degrees occurring in S, has been heavily studied as an invariant of nonunique factorization. In this paper, we investigate the set $\mathsf{C}(S)$ of catenary degrees achieved by elements of S as a factorization invariant, focusing on the case where S in finitely generated (where $\mathsf{C}(S)$ is known to be finite). Answering an open question posed by García-Sánchez, we provide a method to compute the smallest nonzero element of $\mathsf{C}(S)$ that parallels a well-known method of computing the maximum value. We also give several examples demonstrating certain extremal behavior for $\mathsf{C}(S)$, and present some open questions for further study.

1. INTRODUCTION

Nonunique factorization theory aims to classify and quantify the failure of elements of cancellative commutative monoids to factor uniquely into irreducibles [14]. Factorization invariants are arithmetic quantities that measure the failure of a monoid's elements to admit unique factorizations. There are many standard invariants used frequently in the literature to compare factorization behavior between monoids, such as the delta set [3], elasticity [8], and ω-primality [1, 16].

Most factorization invariants assign a value to each monoid element determined by its factorization structure. In examining the behavior of an invariant throughout the whole monoid, one often considers quantities such as the supremum of all values attained at its elements, or the set of all such values. For instance, the elasticity of a monoid element is defined as the ratio of its largest and smallest factorization lengths, and the elasticity of the monoid is simply the supremum of the elasticities of its elements.

This paper concerns the catenary degree (Definition 2.3), a factorization invariant that has been the subject of much recent work [2, 6, 17]. The catenary degree $\mathsf{c}(n)$ of a monoid element $n \in S$ is a nonnegative integer derived from combinatorial properties of the set of factorizations of n. Although much of the literature on the catenary degree focuses on the maximum catenary degree attained within S, some recent papers [4, 5] examine the catenary degees of individual monoid elements. In this paper, we investigate the set $\mathsf{C}(S)$ of catenary degrees occuring within S as a factorization invariant, focusing on the setting where S is finitely generated.

It is known from [7] that $\max \mathsf{C}(S)$, the maximum catenary degree attained within a given monoid S, always occurs at least once within a certain finite class of elements

of S called Betti elements (Definition 2.5). This result was later extended to monoids satisfying a certain weakened Noetherian condition in [18]. The second author of [7] conjectured that when S is finitely generated, the minimum nonzero value of $\mathsf{C}(S)$ also occurs at a Betti element of S.

As the main result of this paper, we prove the aformentioned conjecture (Theorem 3.5). This yields a computable bound on the values occuring in $\mathsf{C}(S)$, as well as a characterization of those finitely gererated monoids S for which $\mathsf{C}(S)$ is minimal. We also give in Example 3.1 a semigroup whose set of catenary degrees has a nonzero element that does not occur at a Betti element, demonstrating that this result need not extend the entire set $\mathsf{C}(S)$.

In Section 4, we give several examples that demonstrate certain extremal behavior of $\mathsf{C}(S)$. In particular, we show that if a monoid S has at least 3 minimal generators, then $|\mathsf{C}(S)|$ cannot be bounded by the number of generators of S (Theorem 4.7). We conclude the paper by giving several directions for future study in Section 5.

2. BACKGROUND

Unless otherwise stated, S denotes a finitely generated cancellative commutative monoid, written additively. By passing from S to the quotient by its unit group (if necessary), we will assume that S is *reduced*, that is, S has no nonzero units.

Definition 2.1. An additive submonoid $S \subset \mathbb{N}$ is a *numerical monoid* is $|\mathbb{N} \setminus S| < \infty$. Any numerical monoid S has a unique generating set that is minimal with respect to containment. If we write $S = \langle n_1, \ldots, n_k \rangle$, it is assumed that the integers $n_1, \ldots, n_k$ comprise the minimal generating set of S.

Definition 2.2. Fix $n \in S$. A *factorization* of n is an expression $n = u_1 + \cdots + u_r$ of n as a sum of irreducible elements $u_1, \ldots, u_r$ of S. If $n_1, \ldots, n_k$ minimally generate S, we often write factorizations of n in the form $n = a_1 n_1 + \cdots + a_k n_k$. Write

$$\mathsf{Z}_S(n) = \{(a_1, \ldots, a_k) : n = a_1 n_1 + \cdots + a_k n_k\} \subset \mathbb{N}^k$$

for the *set of factorizations* of $n \in S$. When there is no ambiguity, we often omit the subscript and write $\mathsf{Z}(n)$. Given $\mathbf{a} \in \mathsf{Z}_S(n)$, we denote by $|\mathbf{a}|$ the number of irreducibles in the factorization $\mathbf{a}$, that is, $|\mathbf{a}| = a_1 + \cdots + a_k$.

Definition 2.3. Fix an element $n \in S$. For $\mathbf{a}, \mathbf{b} \in \mathsf{Z}(n)$, the *greatest common divisor of $\mathbf{a}$ and $\mathbf{b}$* is given by

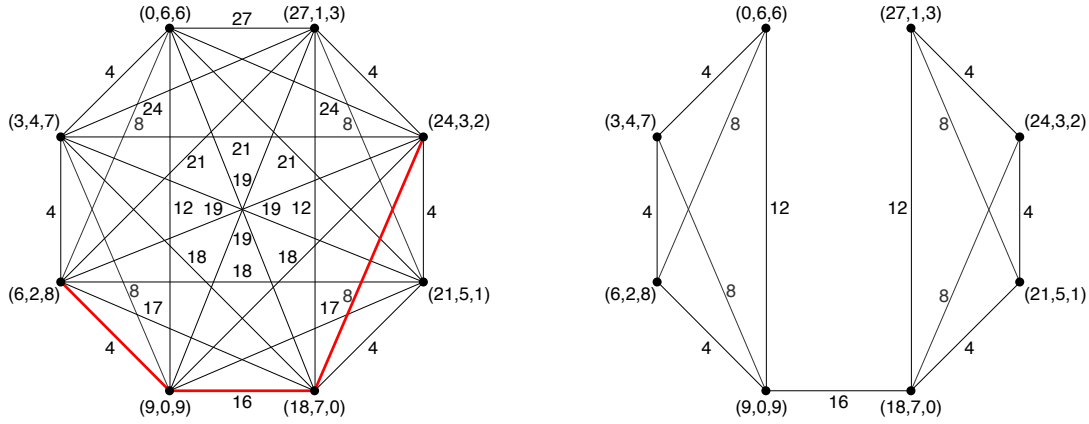$$\gcd(\mathbf{a}, \mathbf{b}) = (\min(a_1, b_1), \ldots, \min(a_r, b_r)) \in \mathbb{N}^r,$$

FIGURE 1. Computing the catenary degree of $450 \in S = \langle 11, 36, 39 \rangle$, as in Example 2.4. Each vertex is labeled with an element of $\mathsf{Z}(450)$, and each edge is labeled with the distance between the factorizations at either end. The diagram on the left depicts all edges, and the diagram on the right includes only those edges labeled at most $\mathsf{c}(450) = 16$. Both graphics were created using the computer algebra system SAGE [19].

and the *distance between* **a** *and* **b** (or the *weight of* (**a**, **b**)) is given by

$$d(\mathbf{a}, \mathbf{b}) = \max(|\mathbf{a} - \gcd(\mathbf{a}, \mathbf{b})|, |\mathbf{b} - \gcd(\mathbf{a}, \mathbf{b})|).$$

Given $\mathbf{a}, \mathbf{b} \in \mathsf{Z}(n)$ and $N \geq 1$, an *N-chain from* **a** *to* **b** is a sequence $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathsf{Z}(n)$ of factorizations of n such that (i) $\mathbf{a}_1 = \mathbf{a}$, (ii) $\mathbf{a}_k = \mathbf{b}$, and (iii) $d(\mathbf{a}_{i-1}, \mathbf{a}_i) \leq N$ for all $i \leq k$. The *catenary degree of n*, denoted $\mathsf{c}(n)$, is the smallest non-negative integer N such that there exists an N-chain between any two factorizations of n. The *set of catenary degrees of S* is the set $\mathsf{C}(S) = \{\mathsf{c}(m) : m \in S\}$.

Example 2.4. Consider the numerical monoid $S = \langle 11, 36, 39 \rangle$. The left-hand picture in Figure 1 depicts the factorizations of $450 \in S$ along with all pairwise distances. There exists a 16-chain between any two factorizations of 450; one such 16-chain between $(6, 2, 8)$ and $(24, 3, 2)$ is depicted with bold red edges. Since every 16-chain between these factorizations contains the edge labeled 16 at the bottom, we have $\mathsf{c}(450) = 16$.

This can also be computed in a different way. In the right-hand picture of Figure 1, only distences of at most 16 are depicted, and the resulting graph is connected. Removing the edge labeled 16 yields a disconnected graph, so we again conclude $\mathsf{c}(450) = 16$.

Factorizations of certain monoid elements, called Betti elements (Definition 2.5), contain much of the structural information used to construct chains of factorizations, and thus are closely related to the catenary degree. For instance, it is known that the maximum catenary degree occuring in a monoid S is guaranteed to occur at a Betti
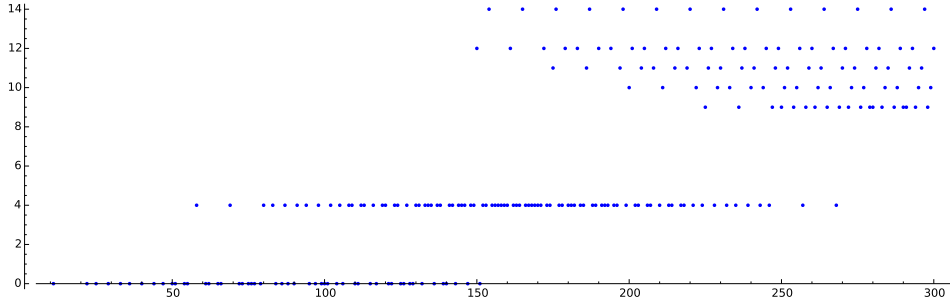
FIGURE 2. A SAGE plot [19] showing the catenary degrees for the numerical monoid $S = \langle 11, 25, 29 \rangle$ discussed in Example 3.1.

element of S (Theorem 2.6). Continuing in this vein, we will show in Corollary 3.7 that the minimum nonzero catenary degree of S also occurs at a Betti element of S.

Definition 2.5. Fix a finitely generated monoid S. For each nonzero $n \in S$, consider the graph ∇_n with vertex set $\mathsf{Z}(n)$ in which two vertices $\mathbf{a}, \mathbf{b} \in \mathsf{Z}(n)$ share an edge if $\gcd(\mathbf{a}, \mathbf{b}) \neq 0$. If ∇_n is not connected, then n is called a *Betti element* of S. We write

$$\mathrm{Betti}(S) = \{b \in S : \nabla_b \text{ is disconnected}\}$$

for the set of Betti elements of S.

Theorem 2.6 ([7, Theorem 3.1]). *For any finitely generated monoid S,*

$$\max \mathsf{C}(S) = \max\{\mathsf{c}(b) : b \in \mathrm{Betti}(S)\}.$$

3. THE MINIMUM NONZERO CATENARY DEGREE

By Theorem 2.6, in order to compute the maximum catenary degree achieved in a monoid S, it suffices to compute the catenary degree of each of its Betti elements. The same cannot be said for all of the values in $\mathsf{C}(S)$; see Example 3.1. The main result of this section is Theorem 3.4, which implies that the minimum nonzero catenary degree, like the maximum catenary degree, achieved in a finitely generated monoid S occurs at a Betti element of S.

Example 3.1. Let $S = \langle 11, 25, 29 \rangle \subset \mathbb{N}$. The catenary degrees of S are plotted in Figure 2. The only Betti elements of S are 58, 150, and 154, which have catenary degrees 4, 12, and 14, respectively. However, $\mathsf{c}(175) = 11$ is distinct from each of these values. However, by Corollary 3.7, every element of S with at least two distinct factorizations has catenary degree at least 4 and at most 14.

Lemma 3.2 and Proposition 3.3 are used in the proof of Theorem 3.4.

Lemma 3.2. *Suppose $S = \langle n_1, \ldots, n_k \rangle$. Fix $n \in S$ with $|\mathsf{Z}(n)| \geq 2$, let B be the set of Betti elements of S that divide n, and let*

$$b = \min\{\mathsf{c}(m) : m \in B\}.$$

For each $\mathbf{a} = (a_1, \ldots, a_k) \in Z(n)$, there exists $\mathbf{a}' \in Z(n)$ such that $d(\mathbf{a}, \mathbf{a}') \geq b$.

Proof. Let $X = \{(x_1, \ldots, x_k) : 0 \leq x_i \leq a_i \text{ for } 1 \leq i \leq k\}$ and let

$$F = \{x \in X : |\mathsf{Z}(x_1 n_1 + \cdots + x_k n_k)| \geq 2\} \subset X.$$

Note that F forms a finite nonempty partially ordered set with unique maximal element $\mathbf{a}$. Choose a minimal element $\mathbf{b} = (b_1, \ldots, b_k) \in F$, and let $m = b_1 n_1 + \ldots + b_k n_k$.

Minimality of $\mathbf{b}$ implies that $|\mathsf{Z}(m - n_i)| = 1$ for each positive b_i, so any factorization $\mathbf{b}' \in \mathsf{Z}(m)$ with $\mathbf{b}' \neq \mathbf{b}$ satisfies $\gcd(a_i', b_i') = 0$. In particular, $m \in B$. Fix $\mathbf{b}' \in \mathsf{Z}(m)$ with $\mathbf{b}' \neq h$, and let $\mathbf{a}' = \mathbf{b}' + \mathbf{a} - \mathbf{b} \in \mathsf{Z}(n)$. We have

$$d(\mathbf{a}, \mathbf{a}') = d(\mathbf{b} + \mathbf{a} - \mathbf{b}, \mathbf{b}' + \mathbf{a}' - \mathbf{b}) = d(\mathbf{b}, \mathbf{b}') \geq \mathsf{c}(m) \geq b,$$

as desired. $\square$

Proposition 3.3. *Suppose $S = \langle n_1, \ldots, n_k \rangle$. Fix $n \in S$ with $|\mathsf{Z}(n)| \geq 2$, let B be the set of Betti elements of S that divide n, and let*

$$b = \min\{\mathsf{c}(m) : m \in B\}.$$

Given distinct $\mathbf{a}, \mathbf{b} \in \mathsf{Z}(n)$ with $d(\mathbf{a}, \mathbf{b}) < b$, there exists $\mathbf{x} \in \mathsf{Z}(n)$ such that

$$\max\{|\mathbf{a}|, |\mathbf{b}|\} < |\mathbf{x}|.$$

Proof. First, suppose $\gcd(\mathbf{a}, \mathbf{b}) = 0$, so that $d(\mathbf{a}, \mathbf{b}) = \max\{|\mathbf{a}|, |\mathbf{b}|\}$. By Lemma 3.2, there exists $\mathbf{x} \in Z(n)$ such that $d(\mathbf{a}, \mathbf{x}) \geq b$. The strict inequality

$$|\mathbf{a} - \gcd(\mathbf{a}, \mathbf{x})| \leq |\mathbf{a}| < b \leq d(\mathbf{a}, \mathbf{x}) = \max\{|\mathbf{a} - \gcd(\mathbf{a}, \mathbf{x})|, |\mathbf{x} - \gcd(\mathbf{a}, \mathbf{x})|\}$$

implies $d(\mathbf{a}, \mathbf{x}) = |\mathbf{x} - \gcd(\mathbf{a}, \mathbf{x})|$. This means

$$\max\{|\mathbf{a}|, |\mathbf{b}|\} = d(\mathbf{a}, \mathbf{b}) < b \leq d(\mathbf{a}, \mathbf{x}) = |\mathbf{x} - \gcd(\mathbf{a}, \mathbf{x})| \leq |\mathbf{x}|,$$

which proves the claim in this case.

Now, suppose $\gcd(\mathbf{a}, \mathbf{b}) \neq 0$. Let $\mathbf{a}' = \mathbf{a} - \gcd(\mathbf{a}, \mathbf{b})$ and $\mathbf{b}' = \mathbf{b} - \gcd(\mathbf{a}, \mathbf{b})$, and fix $n' \in S$ such that $\mathbf{a}', \mathbf{b}' \in \mathsf{Z}(n')$. Since any Betti element dividing n' also divides n, the above argument ensures the existence of $\mathbf{x}' \in \mathsf{Z}(n')$ such that

$$\max\{|\mathbf{a}|, |\mathbf{b}|\} = \max\{|\mathbf{a}'|, |\mathbf{b}'|\} + |\gcd(\mathbf{a}, \mathbf{b})| < |\mathbf{x}'| + |\gcd(\mathbf{a}, \mathbf{b})|.$$

Choosing $\mathbf{x} = \mathbf{x}' + \gcd(\mathbf{a}, \mathbf{b})$ completes the proof. $\square$

Theorem 3.4. *Suppose $S = \langle n_1, \ldots, n_k \rangle$. Fix $n \in S$ with $|\mathsf{Z}(n)| \geq 2$, and let B denote the set of Betti elements of S that divide n. Then*

$$\mathsf{c}(n) \geq \min\{\mathsf{c}(m) : m \in B\}.$$

Proof. Let $b = \min\{\mathsf{c}(m) : m \in B\}$, and let

$$V = \{\mathbf{a} \in \mathsf{Z}(n) : d(\mathbf{a}, \mathbf{b}) < b \text{ for some } \mathbf{b} \in \mathsf{Z}(n)\} \subset \mathsf{Z}(n).$$

If $V = \emptyset$, then $d(\mathbf{a}, \mathbf{b}) \geq b$ for all $\mathbf{a}, \mathbf{b} \in \mathsf{Z}(n)$, and it follows that $\mathsf{c}(n) \geq b$. Otherwise, choose $\mathbf{a} \in V$ such that $|\mathbf{a}|$ is maximal among elements of V. Since $\mathbf{a} \in V$, there exists $\mathbf{b} \in \mathsf{Z}(n)$ such that $d(\mathbf{a}, \mathbf{b}) < b$. By Proposition 3.3, there exists $\mathbf{x} \in \mathsf{Z}(n)$ such that $\max\{|\mathbf{a}|, |\mathbf{b}|\} < |\mathbf{x}|$. Since $|\mathbf{a}| \leq \max\{|\mathbf{a}|, |\mathbf{b}|\} < |\mathbf{x}|$, maximality of $|\mathbf{a}|$ ensures that $\mathbf{x} \notin V$. Consequently, $d(\mathbf{x}, \mathbf{x}') \geq b$ for all $\mathbf{x}' \in Z(n)$ with $\mathbf{x}' \neq \mathbf{x}$, so $\mathsf{c}(n) \geq b$.   □

We conclude this section with several immediate consequences Theorem 3.4. The first is Corollary 3.5, in the spirit of Theorem 2.6.

Corollary 3.5. *If $n \in S$ satisfies $\mathsf{c}(n) > 0$, then*

$$\mathsf{c}(n) \geq \min\{\mathsf{c}(m) : m \in \mathrm{Betti}(S)\}.$$

In particular, $\min \mathsf{C}(S) \setminus \{0\}$ is the catenary degree of some Betti element of S.

Remark 3.6. The proof of Theorem 2.6 given in [7] can be easily extended to show that the catenary degree of any monoid element is bounded above by the catenary degrees of the Betti elements dividing it. We record this in Corollary 3.7.

Corollary 3.7. *Fix $n \in S$ with $\mathsf{c}(n) > 0$, and let B denote the set of Betti elements of S that divide n. Then*

$$\min\{\mathsf{c}(m) : m \in B\} \leq \mathsf{c}(n) \leq \max\{\mathsf{c}(m) : m \in B\}.$$

Lastly, Corollary 3.8 classifies those monoids S for which $|\mathsf{C}(S)|$ is minimal, and generalizes [11, Theorem 19].

Corollary 3.8. $\mathsf{C}(S) = \{0, c\}$ *if and only if $\mathsf{c}(m) = c$ for all $m \in \mathrm{Betti}(S)$.*

Remark 3.9. The set $\mathsf{C}(S)$ of catenary degrees occuring in a monoid S shares many similarities to the delta set $\Delta(S)$. In fact, the maximum element of $\Delta(S)$ is known to lie in the delta set of a Betti element [2]. In contrast, this need not hold for the minimum element of $\Delta(S)$. For example, the numerical monoid $S = \langle 30, 52, 55 \rangle$ has delta set $\Delta(S) = \{1, 2, 3, 5\}$, but its only Betti elements are 260 and 330, and their delta sets are given by $\Delta(260) = \{2\}$ and $\Delta(330) = \{5\}$.

4. Some extremal examples of $\mathsf{C}(S)$

The first results of this section examine the relationship between the number of minimal generators of a monoid S and the cardonality of $\mathsf{C}(S)$. In particular, we exhibit an infinite family of numerical monoids with identical sets of catenary degrees but distinct numbers of minimal generators (Corollary 4.3), and an infinite family of 3-generated numerical monoids whose sets of catenary degrees have distinct cardinalities (Theorem 4.7). Together, these results demonstrate that if $S = \langle n_1, \ldots, n_k \rangle$ for $k \geq 3$, then neither k or $|\mathsf{C}(S)|$ can be bounded in terms of the other. We conclude the

section with Theorem 4.10, which demonstrates that for any $c \geq 3$, there exists a finitely generated monoid whose set of catenary degrees equals $\{0, 2, 3, \ldots, c\}$.

Example 4.1. Suppose $S \subset \mathbb{N}^k$ has two minimal generators. If $k \geq 2$ and the generators of S are linearly independent, then S is factorial, so $\mathsf{C}(S) = \{0\}$. Otherwise, S is isomorphic to a numerical monoid $\langle n_1, n_2 \rangle \subset \mathbb{N}$, and [5, Remark 2.2] implies $\mathsf{C}(S) = \{0, n_2\}$.

Given integers $k \geq 3$ and $c \geq 3$, Corollary 4.3 identifies a k-generated numerical monoid S with catenary degree c whose elements achieve precisely 3 distinct catenary degrees. As this monoid is generated by an arithmetic sequence, the result follows immediately from Theorem 4.2, which appeared in [5].

Theorem 4.2 ([5, Theorem 3.1]). *If $S = \langle a, a+d, \ldots, a+(k-1)d \rangle \subset \mathbb{N}$ with $\gcd(a, d) = 1$ and $3 \leq k \leq a$, then $\mathsf{C}(S) = \{0, 2, \lceil \frac{a}{k-1} \rceil + d\}$.*

Corollary 4.3. *Fix $c \geq 3$ and $k \geq 3$. Let $S = \langle k, k + (c-2), \ldots, k + (k-1)(c-2) \rangle$. Then $\mathsf{C}(S) = \{0, 2, c\}$.*

Remark 4.4. Corollary 3.8 classifies monoids S with exactly one nonzero element in $\mathsf{C}(S)$. Such monoids can also have arbitrarily large minimal generating sets. In particular, if $p_1 < \cdots < p_k$ are k distinct primes, then the numerical monoid

$$S = \langle (p_1 \cdots p_k)/p_k, \ldots, (p_1 \cdots p_k)/p_1 \rangle$$

has a single Betti element, so $\mathsf{C}(S) = \{0, p_k\}$ by Corollary 3.8. See [11] for more detail on this class of numerical monoids.

Theorem 4.7 defines an infinite family of 3-generated numerical monoids, based on a parameter k, whose set of catenary degrees has cardonality at least $k + 1$. Example 3.1 discusses the resulting numerical monoid when $k = 5$. Before we prove the theorem, we recall Lemmas 4.5 and 4.6, whose proofs are given in [12].

Lemma 4.5. *If $S = \langle n_1, n_2 \rangle$ is a numerical monoid and $n \in \mathbb{Z}$, then $n \in S$ if and only if $n_1 n_2 - n_1 - n_2 - n \notin S$.*

Lemma 4.6. *Let $S = \langle n_1, n_2, n_3 \rangle \subset \mathbb{N}$ be a numerical monoid. Each element of $\mathrm{Betti}(S)$ can be written in the form*

$$c_i n_i = r_{ij} n_j + r_{ik} n_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$ and $c_i = \min\{c > 0 : cn_i \in \langle n_j, n_k \rangle\}$.

Theorem 4.7. *Fix $k \geq 3$, and let $n_1 = 2k + 1$, $n_2 = 6k - 5$, $n_3 = 6k - 1$, and $S = \langle n_1, n_2, n_3 \rangle$.*

  (i) The Betti elements of S are $u = (3k - 1)n_1$, $v = (k + 1)n_2$ and $w = 2n_3$.
  (ii) We have $\{4, 3k - 1\}, \{2k - 1, \ldots, 3k - 3\} \subset \mathsf{C}(S)$.

Proof. Notice the generators of S are all pairwise coprime. Fix $a < 3k - 1$, and write $a = 3b + c$ for $0 \le c < 3$. We have

$$
\begin{aligned}
n_2 n_3 - n_2 - n_3 - an_1 &= (6k-5)(6k-1) - (6k-5) - (6k-1) - a(2k+1) \\
&= 36k^2 - (2a+48)k + (11-a) \\
&= (2ck + b - 1)n_2 + (6k - 2ck - 2b + c - 6)n_3,
\end{aligned}
$$

so by Lemma 4.5, $an_1 \notin \langle n_2, n_3 \rangle$. This means $u = (3k-1)n_1 = kn_2 + n_3$ is a Betti element of S by Lemma 4.6. Similarly, for each $a < k + 1$, we have

$$
\begin{aligned}
n_1 n_3 - n_1 - n_3 - an_2 &= 12k^2 - (6a+4)k + (5a-1) \\
&= (3a-1)n_1 + (2k - 2a)n_3,
\end{aligned}
$$

so $v = (k+1)n_2 = (3k-4)n_1 + n_3$ is also a Betti element of S by Lemmas 4.5 and 4.6. Applying Lemmas 4.5 and 4.6 once more, we conclude from

$$
n_1 n_2 - n_1 - n_2 - n_3 = 12k^2 - 18k = (3k-5)n_1 + (k-1)n_2,
$$

that $w = 2n_3 = 3n_1 + n_2$ is the last Betti element of S. This proves (i).

It is easy to check that u, v, and w each have only 2 distinct factorizations, and the first containment of (ii) follows from computing $\mathsf{c}(u) = 3k - 1$ and $\mathsf{c}(w) = 4$. For $0 \le j \le k - 2$, let $s_j = 6k^2 + (6j+1)k - 5j - 5$. We claim each s_j has exactly $j + 2$ distinct factorizations: $s_j = (k + 1 + j)n_2$, which we shall denote by $\mathbf{a}_0 \in \mathsf{Z}(s_j)$ and

$$
s_j = (3k - 1 - 3i)n_1 + (j + 1 - i)n_2 + (2i - 1)n_3
$$

for $1 \le i \le j + 1$, which we shall denote by $\mathbf{a}_i \in \mathsf{Z}(s_j)$. Indeed, this has already been shown for $s_0 = v$ above, and induction on j implies each $s_j = s_{j-1} + n_2$ has exactly $j + 1$ factorizations with at least one copy of n_2. Since $3k - 4 - 3j < n_3$ and $2j + 1 < n_1$, the only factorization of s_j in $\langle n_1, n_3 \rangle$ is $\mathbf{a}_0$, from which the claim follows.

Lastly, if $\mathsf{c}(s_j) = N$, there exists an N-chain eminating from $\mathbf{a}_0$, so

$$
N \ge \min\{d(\mathbf{a}_0, \mathbf{a}_i) : 1 \le i \le j + 1\} = \min\{3k - i - 2 : 1 \le i \le j + 1\} = 3k - 3 - j.
$$

Since $d(\mathbf{a}_i, \mathbf{a}_{i+1}) = 4$ for $1 \le i \le j$, we have $\mathsf{c}(s_j) = 3k - 3 - j$. $\square$

The final result of this section concerns monoids of zero-sum sequences over finite groups (Definition 4.8). Here, we only introduce what is needed to prove Theorem 4.10.

Definition 4.8. Fix a finite group G with $|G| \ge 3$, written additively, and let $\mathcal{F}(G)$ denote the (multiplicatively written) free abelian monoid with basis G. An element $A = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G)$ (called a *sequence* over G) is said to be *zero-sum* if $g_1 + \ldots + g_\ell = 0$ in G. The set $\mathcal{B}(G) \subset \mathcal{F}(G)$ of zero-sum sequences over G is a submonoid of $\mathcal{F}(G)$, called the *block monoid of G*.

Remark 4.9. By [14, Proposition 2.5.6], the block monoid $\mathcal{B}(G)$ of a finite group G is a Krull monoid with class group isomorphic to G and every class contains a prime divisor (we refer the reader to [14] for the background on Krull monoids). The catenary degree of block monoids was recently studied in the context of Krull monoids in [13, 15].

Theorem 4.10. *Let S be the block monoid of a cyclic group G of order $|G| = n \geq 4$, and fix an element $g \in G$ with order $|g| = n$. Then $\mathsf{C}(S) = \{0, 2, 3, \ldots, \mathsf{c}(S)\}$.*

Proof. By [14, Theorem 6.4.7], we have $\mathsf{c}(S) = n$ and hence $\mathsf{C}(S) \subset \{0, 2, 3, \ldots, n\}$. Hence, it remains to show that the interval $[2, n] \subset \mathsf{C}(S)$. First, consider the element
$$A = (2g)^2 g^{2n-4} \in S.$$
The only minimal (that is, irreducible) zero-sum sequences dividing A are given by $U = g^n$, $V = (2g)g^{n-2}$, and $W = (2g)^2 g^{n-4}$. This yields $\mathsf{Z}(A) = \{UW, V^2\}$, from which we conclude that $\mathsf{c}(A) = 2$.

Now, fix $j \in [3, n]$ and let $h = (j-1)g$. Consider the zero-sum sequence
$$A = (-g)^{j-1} g^n h \in S.$$
This time, there are precisely four minimal zero-sum sequences dividing A, namely $U = g^n$, $V = g(-g)$, $W = (-g)^{j-1} h$, and $X = g^{n-j+1} h$. From this, we conclude that $\mathsf{Z}(A) = \{UW, V^{j-1}X\}$, which means in particular that $\mathsf{c}(A) = j$. □

5. FUTURE WORK

The delta set realization problem [9] asks which finite sets $D \subset \mathbb{N}$ satisfy $\Delta(S) = D$ for some monoid S. The results of Section 4 pertain to the realization problem for sets of catenary degrees; we record this here as Problem 5.1.

Problem 5.1. *Fix a finite set $C \subset \mathbb{N}$ such that $C \cap \{0, 1\} = \{0\}$. Does there exist a finitely generated monoid S with $\mathsf{C}(S) = C$?*

In general, it is not easy to prove that a given value c does not equal the catenary degree of any elements of a given monoid S (the same difficulty arises when computing the delta set of a monoid; see Remark 3.9). Computer software can be used to compute the catenary degree of individual elements of S (for instance, the GAP package numericalsgps [10] can do this). However, computing $\mathsf{C}(S)$ via exhaustive search is not possible, and it can be difficult to determine when the whole set $\mathsf{C}(S)$ have been computed. An answer to Problem 5.2 would allow for a more effective use of computer software packages in studying $\mathsf{C}(S)$.

Problem 5.2. *Given a monoid S, determine a (computable) finite class of elements of S on which every catenary degree in $\mathsf{C}(S)$ occurs.*

A *generalized arithmetic sequence* is a sequence of the form
$$a, ah + d, \ldots, ah + (k-1)d$$
for integers a, h, d and k. In [5], a formula is given for the catenary degree of any element in a numerical monoid generated by an arithmetic sequence (from which Theorem 4.2 follows). These results were motivated in part by [17], which computes the maximum catenary degree achieved in any numerical monoid generated by generalized arithmetic sequences. The following natural question arises.

Problem 5.3. *Suppose* $S = \langle a, ah + d, \ldots, ah + (k-1)d \rangle$ *with* $\gcd(a, d) = 1$ *and* $1 < k \leq a$. *Describe the set* $\mathsf{C}(S)$ *in terms of* a, d, h *and* k.

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