

Minimal Zero Sequences of Finite Cyclic Groups

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Let $MZS(G, k)$ denote the set of minimal zero sequences of finite Abelian group G . In this paper we investigate the structure of the elements of this set, and the cardinality of the set itself. We do this for the class of groups $G = \mathbb{Z}_n$ for k both small ($k \leq 4$) and large ($k > \frac{2n}{3}$).

Key Words: Zero-sum problems, minimal zero sequence

1. INTRODUCTION

Let G be a finite Abelian group and $X = \{x_1, x_2, \dots, x_k\}$ a multiset chosen from G . This unordered collection of not necessarily distinct elements of G is traditionally called a *sequence*. We say the *length* of X is k . If $x_1 + x_2 + \dots + x_k = 0$ (in G), then X is called a *zero-sequence*. We denote the set of all zero sequences of G by $ZS(G)$. If X is in $ZS(G)$ but no proper subsequence of X is in $ZS(G)$, then X is called a *minimal zero sequence*. We denote the set of all minimal zero sequences of G of length k by $MZS(G, k)$, and the set of all minimal zero sequences of G of any length by $MZS(G)$. The maximum k for which $MZS(G, k)$ is nonempty is the well-known Davenport constant of G .

Notice that $\text{Aut}(G)$ acts on $ZS(G)$, on $MZS(G)$, and on $MZS(G, k)$, inducing equivalence relations on these sets. We denote by $E(X)$ the set of sequences equivalent to sequence X , as induced in this manner.

We express G canonically as $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_r}$, with $n_1 | n_2 | \dots | n_r$. We say that zero sequence X is *basic* if $E(X)$ contains a zero sequence whose sum in coordinate i is at most n_i (for $1 \leq i \leq r$), when the sum is viewed as an integer. To avoid confusion, henceforth the symbol '+' shall denote addition as integers, and the symbol $\sum X$ shall denote the sum of the elements of X as integers. If G is cyclic, that is its rank $r = 1$, then all basic zero sequences are minimal.

If every element of $MZS(G, k)$ is basic, we say that (G, k) is a *basic pair*, otherwise it is a *non-basic pair*. Chapman, Freeze, and Smith [2] have shown that $(\mathbb{Z}_n, 5)$ is a non-basic pair for all $n \neq 2, 3, 4, 5, 7$; further, for these five values of n , (\mathbb{Z}_n, k) is a basic pair for all k . This left open the question of which (\mathbb{Z}_n, k) are basic pairs.

We offer a partial answer to this question, for all $n \geq 5$ and both very small and large k . We show in Theorems 2.1 and 3.4 that (\mathbb{Z}_n, k) is a basic pair for $k > \frac{2n}{3}$ and $k \leq 3$; whereas $(\mathbb{Z}_n, 4)$ is a non-basic pair if $\gcd(n, 6) \neq 1$.

As an application, we count the number of minimal zero sequences of length greater than $\frac{2n}{3}$.

2. SHORT MINIMAL ZERO SEQUENCES

We first consider the question of whether (\mathbb{Z}_n, k) is a basic pair for $n \leq 4$. We have evidence to support the converse of the second part of the theorem; that is, we believe that if $\gcd(n, 6) = 1$, then $(\mathbb{Z}_n, 4)$ is a basic pair. This has been verified computationally for $n \leq 1000$.

THEOREM 2.1. *Let $n \geq 5$. Then (\mathbb{Z}_n, k) is a basic pair for $k = 1, 2, 3$. If $\gcd(n, 6) \neq 1$, then $(\mathbb{Z}_n, 4)$ is a non-basic pair.*

Proof. The only element of $MZS(\mathbb{Z}_n, 1)$ is $\{0\}$, which is basic. Let $X = \{a, b\} \in MZS(\mathbb{Z}_n, 2)$. It has $a < n$ and $b < n$, and hence $a + b < 2n$, so X is basic. Suppose that $X = \{a, b, c\} \in MZS(\mathbb{Z}_n, 3)$ were non-basic. Then $a + b + c > n$, but $a + b + c < 3n$, so $a + b + c = 2n$. Now, $\phi(y) = n - y$ is an automorphism on $MZS(\mathbb{Z}_n, 3)$, and $\phi(X) = \{n - a, n - b, n - c\}$ has $\sum \phi(X) = (n - a) + (n - b) + (n - c) = 3n - (a + b + c) = 3n - 2n = n$. Hence X is, in fact, basic.

Suppose now that n is even, so $n = 2m$. We will now show that $X = \{1, m, m + 1, 2m - 2\}$ is not basic, and hence that $(\mathbb{Z}_{2m}, 4)$ is a non-basic pair. First, X sums to a multiple of n , but no proper subset does, hence X is a minimal zero sequence. Now, let ϕ be any automorphism of \mathbb{Z}_{2m} . We must have $\phi(y) = ky$, for k some positive odd integer, different from m , less than n . We see that $\phi(X) = \{k, km, km + k, k(2m - 2)\}$. Reducing modulo n , we see that $\phi(X) = \begin{cases} \{k, m, m + k, 2m - 2k\} & \text{if } k < m, \\ \{k, m, k - m, 4m - 2k\} & \text{if } k > m. \end{cases}$

In both cases we have $\sum \phi(X) = 2n$. Hence, X is not basic if n is even.

Now suppose that $3|n$; that is, $n = 3m$. We will now show that $X = \{1, m + 1, 2m + 1, 3m - 3\}$ is not basic, and hence that $(\mathbb{Z}_n, 4)$ is a non-basic pair. First, X sums to a multiple of n , but no proper subset does, hence X is a minimal zero sequence. Now, let ϕ be any automorphism of \mathbb{Z}_n . We must

have $\phi(y) = ky$, for k some positive integer, less than n , relatively prime to n . We have $\phi(X) = \{k, km + k, 2km + k, 3km - 3k\}$. We next note that $\{km + k, 2km + k\}$ are congruent (modulo n) to $\{m + k, 2m + k\}$ in some order, depending on whether $k \equiv 1$ or $k \equiv 2$ (modulo 3). We can now reduce

$$\text{modulo } n, \text{ and find } \phi(X) = \begin{cases} \{k, m + k, 2m + k, 3m - 3k\} & \text{if } k < m, \\ \{k, m + k, k - m, 6m - 3k\} & \text{if } m < k < 2m, \\ \{k, k - 2m, k - m, 9m - 3k\} & \text{if } 2m < k. \end{cases}$$

In all three cases we have $\sum \phi(X) = 2n$. Hence, X is not basic if $3|n$. \blacksquare

3. LONG MINIMAL ZERO SEQUENCES

We now consider minimal zero sequences in \mathbb{Z}_n , long relative to the maximal possible length (namely n). We begin with some structure theorems, and ultimately show that (\mathbb{Z}_n, k) is a basic pair for all $k > \frac{3n-3}{4}$.

We state a theorem that was first proved in [1], was rediscovered in [7], and restated in various forms in [6, 8].

THEOREM 3.1. *Let $k > \frac{n+3}{2}$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X at least $2k - n$ times.*

With a stronger restriction on k , we can get a bit more. This next result has a stronger hypothesis and conclusion than a similar one found in [5]. It has previously appeared in [4], with a substantially different proof.

THEOREM 3.2. *Let $k > \max(\frac{n+3}{2}, \frac{2n}{3})$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X at least $2k - n$ times, whose order is n (in \mathbb{Z}_n).*

Proof. Applying Theorem 3.1, we write $X = \{a^m, b_1, b_2, \dots, b_j\}$ (where m is the multiplicity of a), with $m \geq 2k - n$ and $m + j = k$.

Now, suppose that the order of a were less than n . Then, we can write $a = a'd$ and $n = n'd$, where $\gcd(a', n') = 1$ and $d \geq 2$. However, if $d \geq 3$, we have $n' \leq \frac{n}{3} < m$. Hence X contains n' copies of a , whose sum is $n'a = n'da' = na'$. But this is a proper zero-sum, which is forbidden. Therefore, we must have $d = 2$, n even (since $d|n$), and $m < \frac{n}{2}$ (since a^m is not a zero subsequence). The remainder of the proof develops a contradiction in these circumstances.

We now show that there is an automorphism ϕ of G with $\phi(a) = 2$. Because $\gcd(a', n') = 1$, there is some integer w with $wa' \equiv 1$ modulo n' . If w is odd, then $\gcd(w, n) = 1$ and $\phi(x) = wx$ is the desired automorphism. If w is even, then n' must be odd. In this case, $(w + n')a' \equiv 1$ modulo n' . We have $w + n'$ odd, so $\gcd(w + n', n) = 1$ and therefore $\phi(x) = (w + n')x$

is the desired automorphism. Henceforth we will assume without loss that $a = 2$.

We now consider the odd elements of X . We pair them arbitrarily and take the residue modulo n . The result is $X' = \{2^m, c_1, c_2, \dots, c_{j'}\}$, where some $c_{i'}$ are equal to an even b_i , while others are equal to the reduced sum of two odd elements of X . This is still a minimal zero sequence, and all of its terms are even. Further, we have $j' \geq \frac{j}{2}$. Note that $m + j = k > \frac{2n}{3}$, and hence $j' \geq \frac{j}{2} > \frac{n}{3} - \frac{m}{2} > \frac{n}{3} - \frac{n}{4} + \frac{1}{2} = \frac{n}{12} + \frac{1}{2}$. Therefore, in particular, $j' \geq 2$. Now we will show that any proper subsequence of $\{c_1, c_2, \dots, c_{j'}\}$ has sum at most $n - 2m - 2$, by induction on the cardinality of the subsequence. For the base case, observe that each singleton c_i must have $c_i \leq n - 2m - 2$, as otherwise X' would not be a minimal zero sequence. Now, let S be a proper subsequence. Write $S = S_1 \cup S_2$, the disjoint union of two nonempty subsequences. By the inductive hypothesis, $\sum S_1 \leq n - 2m - 2$ and $\sum S_2 \leq n - 2m - 2$. Adding, we get $\sum S = \sum S_1 + \sum S_2 \leq 2n - 4m - 4 \leq 2n - \frac{4n}{3} - 4 = \frac{2n}{3} - 4 < n$. We have $\sum S$ even, but because S is a proper subsequence, we must not have $\sum S \in [n - 2m, n]$. Therefore $\sum S \leq n - 2m - 2$. Finally, we note that $(c_1 + c_2 + \dots + c_{j'-1}) + c_{j'} \leq n - 2m - 2 + n - 2m - 2 \leq \frac{2n}{3} - 4 < n$. Therefore, because X' is a minimal zero sequence, we must have $2m + c_1 + c_2 + \dots + c_{j'} = n$. However, each c_i is even, so we therefore have the chain of inequalities $n = \frac{n}{3} + \frac{2n}{3} < m + k = 2m + j \leq 2m + 2j' \leq 2m + c_1 + \dots + c_{j'} = n$. This is a contradiction. \blacksquare

COROLLARY 3.1. *Let $n \geq 10, k > \frac{2n}{3}$, and let $X \in MZS(\mathbb{Z}_n, k)$. Then there is some element $a \in \mathbb{Z}_n$ that appears in X more than $\frac{k}{2}$ times, whose order is n (in \mathbb{Z}_n).*

Proof. The condition $n \geq 10$ ensures that $\frac{2n}{3} \geq \frac{n+3}{2}$, so that the conditions of Theorem 3.2 are met. As before, we write $X = \{a^m, b_1, b_2, \dots, b_j\}$. Since $k > \frac{2n}{3}$, we must have $m > 2(\frac{2n}{3}) - n = \frac{n}{3}$. We also have $m + j = k$, and hence $m \geq 2k - n = (m + j) + k - n$. Rearranging, we get $j \leq n - k < \frac{n}{3}$. Combining these two facts, we get $j < \frac{n}{3} < m$, and hence $m > \frac{k}{2}$. \blacksquare

This allows us to conclude that all sufficiently long minimal zero sequences of \mathbb{Z}_n are basic.

THEOREM 3.3. *Let $n \geq 10, k > \frac{3n-3}{4}$. Then $MZS(\mathbb{Z}_n, k)$ is a basic pair.*

Proof. Let $Y \in MZS(\mathbb{Z}_n, k)$. By Theorem 3.2 and Corollary 3.1, there is some element $y \in Y$, of order n , that appears at least $2k - n$ times. Let $\phi \in \text{Aut}(\mathbb{Z}_n)$ be such that $\phi(y) = 1$. Let $X = \phi(Y)$. We will show that $\sum X = n$, which proves the theorem. Write $X = \{1^m, x_1, x_2, \dots, x_j\}$,

where $m \geq 2k - n$, $m + j = k$, and each $x_i > 1$. First, note that if $j = 1$ then $\sum X = m + x_j < m + n < 2n$, so $\sum X = n$. Otherwise, $j > 1$ and we see that each $x_i \leq n - m - 1$, since otherwise X would properly contain a zero sequence. Now, $x_1 < n - m$, but $x_1 + x_2 + \cdots + x_j \geq n - m$. Let w be such that $x_1 + x_2 + \cdots + x_{w-1} < n - m$, but $x_1 + x_2 + \cdots + x_w \geq n - m$. If $w = j$, then because $x_w < n$, we have $x_1 + x_2 + \cdots + x_w = n - m$ and hence $\sum X = n$. Otherwise, $x_1 + x_2 + \cdots + x_w \geq n + 1$ because X is a minimal zero sequence. Subtracting, we get $x_w \geq m + 2$. However, $n - m - 1 \geq x_w \geq m + 2$. Rearranging, we get $m \leq \frac{n-3}{2}$. But also $m \geq 2k - n \geq 2\frac{3n-3}{4} - n = \frac{n-3}{2}$. This is impossible, and hence $w = j$ and thus $\sum X = n$. \blacksquare

It has come to our attention that a stronger result, with a different proof, has been published in [4]:

THEOREM 3.4. *Let $n \geq 10, k > \frac{2n}{3}$. Then $MZS(\mathbb{Z}_n, k)$ is a basic pair.*

4. COUNTING MINIMAL ZERO SEQUENCES

The cardinality of $MZS(\mathbb{Z}_n, k)$ has already been computed for small k , in [3], as follows.

THEOREM 4.1. $|MZS(\mathbb{Z}_n, 2)| = \lfloor \frac{n}{2} \rfloor$. $|MZS(\mathbb{Z}_n, 3)| = \frac{1}{6}(n^2 - \alpha)$, where α is given by:

$$\begin{array}{c|ccccc} (n \bmod 6) \equiv & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \alpha = & 0 & 1 & 4 & -3 & 4 & 1 \end{array}$$

We can find $|MZS(\mathbb{Z}_n, k)|$ for large k with the results of Section 3. For this purpose, we need the following structure theorem.

THEOREM 4.2. *Let $n \geq 10, k > \frac{2n}{3}$, and let $X \in MZS(\mathbb{Z}_n, k)$ be basic. Then there is exactly one $Y \in E(X)$ with $\sum Y = n$.*

Proof. As X is basic, so at least one such Y exists. Suppose Y has i terms of 1, and the remaining $k - i$ terms are not. Hence $n = \sum Y \geq i + 2(k - i) = 2k - i$. Hence $i \geq 2k - n > 2k - \frac{3}{2}k = \frac{k}{2}$. Hence over half of the terms of Y are 1. Suppose that there are $Y, Y' \in E(X)$ with $\sum Y = \sum Y' = n$. Let $\phi \in \text{Aut}(\mathbb{Z}_n)$ with $\phi(Y) = Y'$. By the previous, 1 appears in each more than $\frac{k}{2}$ times. Both 1, $\phi(1)$ appear more than $|Y'|/2$ times in Y' , but there are not enough elements in Y' for these to be different. Hence $\phi(1) = 1$, and therefore ϕ is the identity and $Y = Y'$. \blacksquare

We are now ready to count all minimal zero sequences of sufficiently large length. Computational evidence suggests that the condition $k > \frac{2n}{3}$ can be improved to $k \geq \frac{n+4}{2}$.

THEOREM 4.3. *Let $n \geq 10, k > \frac{2n}{3}$. Then $|MZS(\mathbb{Z}_n, k)| = \phi(n)p_k(n)$, where ϕ is Euler's totient function and $p_k(n)$ denotes the number of partitions of n into k parts.*

Proof. By Theorem 3.4, every minimal zero sequence is basic. Therefore, each equivalence class induced by $Aut(\mathbb{Z}_n)$ includes an element whose sum is n . By Theorem 4.2, each equivalence class contains exactly one element whose sum is n . It is clear that the set of minimal zero sequences whose sum is n is exactly the set of partitions of n into k parts. There are therefore $p_k(n)$ equivalence classes. The cardinality of each equivalence class is $|Aut(\mathbb{Z}_n)| = \phi(n)$. ■

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