Diversity in monoids

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1 Introduction

Abstract

Let M be a (commutative cancellative) monoid. A nonunit element $q \in M$ is called almost primary if for all $a, b \in M$, $q|ab$ implies that there exists $k \in \mathbb{N}$ such that $q|a^k$ or $q|b^k$. We introduce a new monoid invariant, diversity, which generalizes this almost primary property. This invariant is developed and contextualized with other monoid invariants. It naturally leads to two additional properties (homogeneity and strong homogeneity) that measure how far an almost primary element is from being primary. Finally, as an application the authors consider factorizations into almost primary elements, which generalizes the established notion of factorization into primary elements.

Throughout this paper, all monoids under consideration are commutative, and (unless otherwise stated) cancellative, and multiplicative, with identity denoted by 1. If M is a monoid, then $M^\times$ denotes the set of units (or invertible elements) of M. If $\pi \in M \setminus M^\times$, we say that π is an **atom** (or irreducible element) of M if for all $a, b \in M$, $\pi = ab$ implies that $a \in M^\times$ or $b \in M^\times$. The set of atoms of a monoid M is denoted by $\mathcal{A}(M)$. We say that M is an **atomic** monoid if every nonunit of M can be written as a product of atoms. If $S = \{s_1, s_2, \cdots, s_k\}$ is a finite subset of M, then we denote the product of the elements of S by $\prod S := s_1 s_2 \cdots s_k$. If $A \subseteq M \setminus M^\times$, we denote the monoid generated by A to be $[A]$, and we say that A is a **generating set** of M if $M = [A]$. Thus, M is atomic if and only if $M = [\mathcal{A}(M) \cup M^\times]$. By $\mathbb{N}$, $\mathbb{N}_0$, and S_n we mean the set of natural numbers, nonnegative integers, and the group of permutations on n letters, respectively.

Let M be a monoid. We establish some terminology regarding ideal-theoretic properties of M (proofs of the following claims can be found in [6]). If A and B are (non empty) subsets of M, then $AB := \{ab \mid a \in A, b \in B\}$ and if $x \in M$, we denote $\{x\}A$ by xA. A subset I of M is called an **ideal** of M if $IM = I$.[1] If I is an ideal of M, then $I = M$ if and only if $I \cap M^\times \neq \emptyset$. If $I \neq M$, we say that I is a **proper** ideal of M. We call I a **prime ideal** of M if I is a proper ideal and $M \setminus I$ is a submonoid of M (equivalently, for all $a, b \in M$ with $ab \in I$, we have $a \in I$ or $b \in I$). A proper ideal I is a **primary ideal** of M if and only if for all $a, b \in M$ with $ab \in I$, either $a \in I$ or $b^k \in I$ for some k (or, equivalently, if $ab \in I$, then either $a \in I$, $b \in I$ or there exist $m, n \in \mathbb{N}$ such that $a^m \in I$ and $b^n \in I$). We say that I is an **almost primary** ideal if for all $a, b \in M$, $ab \in I$ implies that for some $n \in \mathbb{N}$, $a^n \in I$ or $b^n \in I$. Every prime ideal is primary, and every primary ideal is almost primary, but neither converse holds.

If I is an ideal of M, then the radical of I is

$$\sqrt{I} := \{x \in M \mid x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

As in the case for ideals of a ring, it can be shown that the radical of a primary ideal is prime, and that if I and J are ideals, then $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$. It is easy to see that I is almost primary if and only if $\sqrt{I}$ is a prime ideal.

If $p \in M$, then it is apparent that p is prime if and only if pM is a prime ideal of M. If $q \in M$, we say that q is a **primary element** of M if qM is a primary ideal of M, and q is an **almost primary element** of M if qM is an almost primary ideal of M.

Following Halter-Koch in [5], we say that M is a **weakly factorial monoid** (or WFM) if every nonunit element of M can be written as a product of primary elements. WFMs were named analogously after the weakly factorial domains introduced by Anderson and Mahaney in [1]. If M is a WFM, and if x is a nonunit of M, then (up to associates) there is only one factorization of x into primary elements with mutually distinct radicals (such a factorization is called a reduced factorization–see Section 4 for more details).

In this paper, we define and study a new type of monoid invariant, called diversity. If M is an atomic monoid, if we pick $x, y \in M \setminus M^\times$, and if we write $y = s_1^{a_1} s_2^{a_2} \cdots s_k^{a_k}$ with $s_i \in M$ and $a_i \in \mathbb{N}$, then x divides a power of y if and only if x divides a power of $s_1 s_2 \cdots s_k$. So, to measure how "far" x is from being almost primary, we can look for the largest value of n such that x divides a power of $s_1 s_2 \cdots s_k$, but not a power of any subproduct. This is what we will call the diversity of x.

Definition 1.1. Let M be a monoid.

1. We say that $x|S$ (in M) if $x \in M$, S is a finite subset of M, and if there exists $t \in \mathbb{N}$ such that $x| \left(\prod S\right)^t$.

2. We say that x strictly divides S, denoted $x \| S$ if $x|S$, but $x \nmid T$ for all $T \subsetneq S$.

[1]Our notion of an ideal of a monoid is technically the concept of an s-ideal as defined in [6].

3. We define the **diversity** of x, denoted $\operatorname{div}(x)$, to be

$$\operatorname{div}(x) = \sup\{|S| : S \subseteq M \text{ with } x \| S\}$$

4. We define the **diversity** of M and the **atomic diversity** of M, denoted by $\operatorname{div}(M)$ and $\operatorname{div}_{\mathrm{a}}(M)$, respectively, by

$$\operatorname{div}(M) = \sup_{x \in M} \operatorname{div}(x), \text{ and } \operatorname{div}_{\mathrm{a}}(M) = \sup_{x \in \mathcal{A}(M)} \operatorname{div}(x).$$

If x is a unit, then $x \| S$ for any $S \subseteq M$ with $|S| = 1$. Otherwise, $x \| S$ implies that the elements of S are pairwise nonassociate nonunits.

In Section 2, present some preliminary results, including that, for a nonunit $x \in M$, $\operatorname{div}(x) = 1$ if and only if x is almost primary (Proposition 2.3). Further, if M is atomic and if $x \in M$ is a nonunit, we need only count sets S of atoms with $x||S$ to determine $\operatorname{div}(x)$ (Corollary 2.5). We also show that $\operatorname{div}(x)$ is bounded above by both the tame degree $t(M, x)$ and $\omega(M, x)$, and that for v-Noetherian monoids (in particular, for the multiplicative monoids of Notherian or Krull domains), the diversity of every element is finite.

In Section 3, we introduce two additional properties, called "homogeneous" and "strongly homogeneous" that lie between "almost primary" and "primary" and that are also related to Definition 1.1. We show that all nonunit divisors of homogeneous (respectively strongly homogeneous) elements are themselves homogeneous (respectively strongly homogeneous) (Theorem 3.8) – a property that is not shared by almost primary elements. An element x is homogeneous precisely when $\sqrt{xM}$ is not only a prime ideal, but maximal amongst radicals of principal ideals (Theorem 3.8). We also show that $\operatorname{div}(x)$ is determined if x divides a set of strongly homogeneous elements (Corollary 3.10).

Finally, in Section 4, we consider factorizations of elements into almost primary elements. We find that such factorizations need not be unique; however, they are unique up to length and radical (Proposition 4.3). Factorizations into homogeneous elements are unique precisely when the homogeneous elements in question are primary (Theorem 4.5). Also, we show that every nonunit element of M can be factored into almost primary elements if and only if for every nonunit $x \in M$ with $\operatorname{div}(x) \geq 2$, there exist nonunit $y, z \in M$ such that $x = yz$ and $\operatorname{div}(x) = \operatorname{div}(y) + \operatorname{div}(z)$ (or, in other words, $\operatorname{div}(\cdot) : M \setminus M^{\times} \to \mathbb{N}_0^+$ is as close to a semigroup homomorphism as we can hope–cf. Theorem 4.4).

2 Preliminary results

Proposition 2.1. *Let M be a monoid and let $x, y \in M$. Then $div(xy) \leq div(x) + div(y)$.*

Proof. Let $S \subseteq M$ with $xy \| S$. There exist subsets $S_x, S_y \subseteq S$ such that $x \| S_x$ and $y \| S_y$. Since $xy | S_x \cup S_y$, we must have $S = S_x \cup S_y$. Therefore,

$$\operatorname{div}(x) + \operatorname{div}(y) \geq |S_x| + |S_y| \geq |S_x \cup S_y| = |S|,$$

and $\text{div}(xy) \le \text{div}(x) + \text{div}(y)$. □

We rarely have equality in Proposition 2.1; in fact, for $x \in M$ and $n \in \mathbb{N}$, $\text{div}(x^n) = \text{div}(x) < n \cdot \text{div}(x)$, as shown in the following lemma.

Lemma 2.2. *Suppose $S = \{s_1, s_2, \cdots, s_k\}$ and $x \| S$. Then:*

1. *$x|S$ if and only if $\sqrt{s_1M} \cap \sqrt{s_2M} \cap \cdots \cap \sqrt{s_kM} \subseteq \sqrt{xM}$.*
2. *$x\|S$ if and only if both $\sqrt{s_1M} \cap \sqrt{s_2M} \cap \cdots \cap \sqrt{s_kM} \subseteq \sqrt{xM}$ and if no $\sqrt{s_iM}$ can be omitted from the intersection for $1 \le i \le k$.*
3. *If $x\|S$, then $\sqrt{s_iM}$ and $\sqrt{s_jM}$ are incomparable for each $i \neq j$.*
4. *For all $m \in \mathbb{N}$, $x\|S$ if and only if $x^m\|S$. Consequently, $div(x) = div(x^m)$.*

Proof. We have $x|\{s_1, s_2, \cdots, s_k\}$ if and only if $xr = (s_1s_2\cdots s_k)^t$ for some $r \in M$ and $t \in \mathbb{N}$, which is equivalent to $s_1s_2\cdots s_k \in \sqrt{xM}$, which is also equivalent to

$$\sqrt{s_1s_2\cdots s_kM} = \sqrt{s_1M} \cap \sqrt{s_2M} \cap \cdots \cap \sqrt{s_kM} \subseteq \sqrt{xM}.$$

Thus, statement 1 is proved.

We see that 2 follows from 1 and the fact that $\sqrt{s_iM}$ can be omitted (for some i) if and only if $x|S \setminus \{s_i\}$.

We see that 3 follows directly from 2.

Finally, 4 follows from 2, the fact that $\sqrt{xM} = \sqrt{x^mM}$, and the definition of diversity. □

Proposition 2.3. *Let M be a monoid and let $x \in M$. Then:*

1. *If $x\|S$, then the elements of S are pairwise nonassociate, and S contains at most one unit. Further, S contains a unit only if $x \in M^\times$.*
2. *$div(x) = 1$ if and only if either $x \in M^\times$ or x is almost primary.*
3. *If $x\|S$ and $y \in S$, then neither x nor y divides $S \setminus \{y\}$.*
4. *If $x\|S$ and $\{y, z\} \subseteq S$, then $x\|S \cup \{yz\} \setminus \{y, z\}$.*
5. *If $x|R$ and $(\prod R)|S$, then $x|S$.*

Proof. We will prove 2; the remaining parts are straightforward.

First, if $x \in M^\times$, then for all $y \in M$, $x|\{y\}$, and hence $\text{div}(x) = 1$.

So, assume that x is almost primary, and suppose that $x\|S$. Write $S = \{s_1, s_2, \cdots, s_k\}$, and pick $r \in M$ and $t \in \mathbb{N}$ such that $xr = (s_1s_2\cdots s_k)^t = s_1^ts_2^t\cdots s_k^t$. Since x is almost primary, we have $x|(s_i^t)^m$ for some $i, m \in \mathbb{N}$. But then $x|\{s_i\}$, implying that $k = 1$ and $\text{div}(x) = 1$.

On the other hand, suppose that $\text{div}(x) = 1$ and that $x \notin M^\times$. If we have $a, b \in M$ with $x|ab$, then $x|\{a, b\}$, therefore $x|\{a\}$ or $x|\{b\}$, and hence $x|a^m$ or $x|b^m$ for some $m \in \mathbb{N}$. Therefore x is almost primary. □

Theorem 2.4. *Let M be a monoid, and let A be a generating set of M. Then, for all $x \in M$, div$(x) = \sup\{|S| : S \subseteq A$ with $x \| S\}$.*

Proof. Set $\alpha(x) = \sup\{|S| : S \subseteq A \text{ with } x\|S\}$. By Definition 1.1, $\operatorname{div}(x) \geq \alpha(x)$, and if $x \in M^\times$, then $\operatorname{div}(x) = \alpha(x) = 1$. Suppose now that $x \notin M^\times$. Now choose $S = \{s_1, s_2, \ldots, s_k\} \subseteq M$ with $x\|S$. Proposition 2.3 yields that $S \cap M^\times$ is empty. For each i for $1 \leq i \leq k$, write $s_i = a_{i1}a_{i2}\cdots a_{in_i}$, where $n_i \in \mathbb{N}$ and each $a_{ij} \in A$. Then, setting $T = \{a_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq n_i\}$, we see that $x|T$. Therefore there exists $U \subseteq T$ such that $x\|U$.

We claim that for each i with $1 \leq i \leq m$, there exists some j with $1 \leq j \leq n_i$ such that $a_{ij} \in U$. To see why this claim is true, suppose (without loss of generality) that $a_{1j} \notin U$ for all $1 \leq j \leq n_1$. Then, $x|\{a_{ij} \mid 2 \leq i \leq k, 1 \leq j \leq n_i\}$, implying that $x|\{s_2, s_3, \cdots, s_k\}$, a contradiction. Therefore $\alpha(x) \geq |U| \geq k$. By considering all such S, we have $\alpha(x) \geq \operatorname{div}(x)$ and hence $\operatorname{div}(x) = \alpha(x)$. □

Corollary 2.5. *Let M be an atomic monoid. Then, for all $x \in M \setminus M^\times$, div$(x) = \sup\{|S| : S \subseteq \mathcal{A}(M)$ with $x\|S\}$.*

Proof. Since M is atomic, $M = [\mathcal{A}(M) \cup M^\times]$. We combine Theorem 2.4 and Proposition 2.3.1. □

We now relate diversity to some other monoid invariants, beginning with the ω invariant introduced in [2].

Definition 2.6. Let M be a monoid. For $a, b \in M$, let $\omega(a, b)$ denote the smallest $N \in \mathbb{N} \cup \{\infty\}$ with the following property: for all $n \in \mathbb{N}$ and $a_1, a_2, \cdots, a_n \in M$, if $a = a_1a_2\cdots a_n$ and if $b|a$, then there exists a subset $\Omega \subseteq \{1, 2, \cdots, n\}$ such that $|\Omega| \leq N$ and $b|\prod_{i\in\Omega} a_i$. For $b \in M$, we define

$$\omega(M, b) = \sup\{\omega(a, b) \mid a \in M\} \in \mathbb{N} \cup \{\infty\}.$$

Proposition 2.7. *Let M be a monoid, and let $x \in M$. Then div$(x) \leq \omega(M, x)$.*

Proof. Let $x\|\{s_1, s_2, \cdots, s_k\}$. Then there exists $t \in \mathbb{N}$ and $r \in M$ such that $xr = (s_1s_2\cdots s_k)^t$. Therefore x divides a product of kt elements of M. If $\omega(M, x) < k$, then x would divide a proper subset of $\{s_1, s_2, \cdots, s_k\}$, a contradiction. Thus, $\omega(M, x) \geq k$, implying that $\operatorname{div}(x) \leq \omega(M, x)$. □

It should be noted that if M is a v-Noetherian monoid, as defined in [3] (in particular, the multiplicative monoid of a Noetherian or Krull domain is a v-Noetherian monoid), then $\omega(M, x) < \infty$ for all $x \in M$ (cf. Lemma 3.5 of [4]). Also, if M is atomic and if π is a non-prime atom of M, then $\omega(M, \pi) \leq t(M, \pi)$, where $t(M, \pi)$ denotes the tame degree of M with respect to π, as defined in [3]. This proves the following corollary.

Corollary 2.8. *Let M be an atomic monoid. Then:*

1. *For every non-prime atom $\pi \in M$, div$(\pi) \leq t(M, \pi)$.*

2. *If M is not factorial, then $div_a(M) \leq t(M, \mathcal{A}(M))$, where* $t(M, \mathcal{A}(M)) = \sup\{t(M, \pi) : \pi \in \mathcal{A}(M)\}$.

3. *If M is a v-Noetherian monoid, then $div(x) < \infty$ for every $x \in M$.*

Note that if π is a prime element of M, then $t(M, \pi) = 0$ and $\text{div}(\pi) = 1$. To show that the hypothesis of Corollary 2.8.3 is necessary, we produce an example of a monoid with an element with infinite diversity.

Example 2.9. Let A be a countably infinite set with $A = \{x\} \cup \{y_{ij} \,|\, i, j \in \mathbb{N}_0, 0 \leq j \leq i\}$. Let F be the free monoid on this set, and let M be the monoid that results from modding F out by the following relations for all $n \in \mathbb{N}$:

$$x^{n+1} = y_{n0} y_{n1} \cdots y_{nn}.$$

Then, $A = \mathcal{A}(M)$ and M is atomic and half-factorial. However, for each $n \in \mathbb{N}$, we have $x \| \{y_{n0}, y_{n1}, \cdots, y_{nn}\}$. Therefore $\text{div}(x) = \infty$ (and M is not v-Noetherian).

The next two examples show that the diversity of a monoid is, in general, independent of the catenary degree (as defined in [3]). Also, the first example shows that we need not have equality in Proposition 2.7.

Example 2.10. Consider the following multiplicative submonoid of $\mathbb{N}$, known as the Hilbert monoid: $H = 1 + 4\mathbb{N}_0$. The atoms of H are either rational primes congruent to 1 mod 4 (these atoms are prime in H) or of the form pq where p and q are rational primes congruent to 3 mod 4 (these are the non-prime atoms of H). Given an atom pq of the latter type, it is easy to see that $\text{div}(pq)$ is 2 if $p \neq q$, and 1 otherwise. Therefore $\text{div}_\text{a}(H) = 2$.

Given distinct rational primes $p_1, p_2, \cdots, p_{2n}$, each congruent to 3 mod 4, we have $p_1 p_2 \cdots p_{2n} \| \{p_1^2, p_2^2, \cdots, p_{2n}^2\}$, whence $\text{div}(H) = \infty$. However $c(H)$, the catenary degree of H, is 2 (cf. [3]).

Further, if p is a rational prime congruent to 3 mod 4, then it is routine to check that p^2 is almost primary. If q is a rational prime other than p that is congruent to 3 mod 4, then $p^2 | (pq)(pq)$, but $p^2 \nmid pq$. Therefore $\omega(H, p^2) \geq 2 > 1 = \text{div}(p^2)$.

Example 2.11. Let $M = [2, 3]$ be the additive submonoid of $\mathbb{N}_0$ generated by 2 and 3. For all $x \in M$, $x | \{3\}$, thus $\text{div}(M) = 1$. However, $c(M) = 3$ (cf. [3]).

3 Homogeneous and strongly homogeneous elements

Although diversity, as an invariant, cannot differentiate among prime, primary, or almost primary elements, it can differentiate between almost primary and primary elements by the following criterion: if $x, y \in M \setminus M^\times$ with y primary, then $x | \{y\}$ implies that $y | \{x\}$. However, such symmetry need not hold for almost primary elements.

Example 3.1. Let $M = \{2^a 3^b : a \in \mathbb{N}, b \in \mathbb{N}_0\} \cup \{1\}$ be a multiplicative submonoid of $\mathbb{N}$. For any $x \in M$, $x|\{6\}$, whence $\mathrm{div}(M) = 1$. Also, $2|\{6\}$, but $6 \nmid \{2\}$, as no power of 2 is a multiple (in $\mathbb{N}$) of 3.

The ability of y to divide $\{x\}$ whenever x divides $\{y\}$ will be of great concern to us, so we establish some notation concerning this relation. This relation was previously used by Halter-Koch for primary elements in [5].

Definition 3.2. Let M be a monoid and let $x, y \in M \setminus M^\times$. We say that y is **related** to x, denoted by $y \sim x$, if $x|\{y\}$.

Clearly, $y \sim x$ if and only if $\sqrt{yM} \subseteq \sqrt{xM}$. Also, it is easy to see that $\sim$ is a reflexive and transitive relation on $M \setminus M^\times$. As noted above, $\sim$ need not be symmetric. We wish not only to study nonunits for which $\sim$ is symmetric, but to also generalize this notion.

Definition 3.3. Let M be a monoid, and let $x \in M \setminus M^\times$.

1. We say that x is **homogeneous** if $\mathrm{div}(x) = 1$ and if for all $y \in M \setminus M^\times$, $y|\{x\}$ implies that $x|\{y\}$ (or, equivalently, if $x \sim y$, then $y \sim x$).
2. We say that x is **strongly homogeneous** if $\mathrm{div}(x) = 1$ and if for all $y \in M \setminus M^\times$ and $S \subseteq M$ with $x \in S$, we have $y\|S$ implies $x|\{y\}$.

Corollary 3.4. *Let M be a monoid. The relation $\sim$ is an equivalence relation on the set of homogeneous elements of M.*

Proposition 3.5. *Let M be a monoid and $x \in M$ a nonunit. Then the following implications hold for properties of x:*
primary $\Rightarrow$ strongly homogeneous $\Rightarrow$ homogeneous $\Rightarrow$ almost primary

Proof. Let x be primary and pick $y \in M \setminus M^\times$ and $S \subseteq M$ with $x \in S$ and $y\|S$. Then there exists $t \in \mathbb{N}$ and $r \in M$ with $x^t|yr$. As x^t is primary and $x^t \nmid r$ (or else $y|S \setminus \{x\}$), we see that $x^t|y^m$ for some $m \in \mathbb{N}$. Therefore $x|\{y\}$ and x is strongly homogeneous. The other implications are clear. □

We now give examples to show that none of the implications above are reversible. In Example 3.1 above, the element 6 is almost primary but not homogeneous.

Example 3.6. (*An example of a strongly homogeneous element that is not primary.*) Let $M = \mathbb{Z} \setminus \{0, -1\}$ (under multiplication). The atoms of M are of the form $\pm p$, where p is a prime natural number. Note that no atom of M is primary. To see why, if p and q are rational primes with $|p| \neq |q|$, then $p|(-p)q$. However, $p \nmid -p$ and p divides no power of q, implying that p is not primary.

We will now show that every atom of M is strongly homogeneous. Let $p \in M$ be an atom, and suppose that $p|ab$. Then, without loss of generality, we have $p|a$ in $\mathbb{N}$, hence $p|a^2$ in M, and p is almost primary.

Suppose now we have $y \in M \setminus M^\times$ with $y\|S$ and $p \in S$. Choose $r \in M$ and $t \in \mathbb{N}$ such that $yr = (\prod S)^t$. We have $p^t|yr$. As above, if $p|y$ in $\mathbb{N}$, then $p|y^2$ in

M, hence $p|\{y\}$. However, if $p \nmid y$ in $\mathbb{N}$, then $p^t|r$ in $\mathbb{N}$. The only way to avoid p^t dividing r in M (and hence $y|S \setminus \{p\}$) is for $r = -p^t$. But then $r^2 = p^{2t}$, implying that y^2 – and hence y – divides $S \setminus \{p\}$.

Therefore every atom of M is strongly homogeneous, but not primary.

Example 3.7. (*An example of an element that is homogeneous, but not strongly homogeneous.*) Consider the following multiplicative submonoid of $\mathbb{N}$:

$$M = [\{p_1p_2 \,|\, p_1, p_2 \in \mathbb{N} \text{ are distinct odd primes}\}] \cup 6\mathbb{N}.$$

First, we observe that M contains no power of any rational prime. With this, we will show that 6 is homogeneous, but not strongly homogeneous. If $6|ab$ (for $a, b \in M$), then (without loss of generality) a is even, hence 6 divides a in $\mathbb{N}$. Writing $a = 6m$ ($m \in \mathbb{N}$), we see that $a^2 = 6(6m^2)$ and $6|a^2$ (in M), implying that 6 is almost primary.

Also, if $y \in M \setminus M^\times$ with $y|\{6\}$, then pick $r \in M$, $t \in \mathbb{N}$ such that $yr = 6^t$. As above, if y is even, then $6|\{y\}$. If y is odd, then, in $\mathbb{N}$, we must have $2^t|r$, and it must follow that, in $\mathbb{N}$, $y|3^t$. However, y is then a power of 3, a contradiction. Therefore $6|\{y\}$ and 6 is homogeneous.

To see why 6 is not strongly homogeneous, note that $15|\{6, 35\}$, since $(6 \cdot 35)^2 = 15 \cdot (6 \cdot 490)$. However, $15 \nmid \{6\}$ (as no power of 6 is a multiple, in $\mathbb{N}$, of 5) and $15 \nmid \{35\}$ (as no power of 35 can be a multiple, in $\mathbb{N}$, of 3). Thus $15\|\{6, 35\}$. However, $6 \nmid \{15\}$ (as no power of 15 is even). Therefore 6 is not strongly homogeneous.

Theorem 3.8. *Let M be a monoid, let $x \in M \setminus M^\times$. Then:*

1. *For all $y \in M$, $x \sim y$ if and only if $\sqrt{xM} \subseteq \sqrt{yM}$.*
2. *x is homogeneous if and only if $\sqrt{xM}$ is both a prime ideal and maximal amongst radicals of proper principal ideals.*
3. *x is strongly homogeneous if and only if $div(x) = 1$ and for all $y, z \in M \setminus M^\times$, we have $y\|\{x, z\}$ implies $x|\{y\}$.*
4. *If x is homogeneous (resp. strongly homogeneous), then every nonunit divisor of x is homogeneous (resp. strongly homogeneous).*
5. *If M is atomic with $div_a(M) = 1$, then x is homogeneous if and only if x is strongly homogeneous.*

Proof. 1. This follows from the definitions.

2. If x is homogeneous, then x is almost primary, whence $\sqrt{xM}$ is prime. If $\sqrt{xM} \subseteq \sqrt{yM}$ for some $y \in M \setminus M^\times$, then $x \in \sqrt{yM}$ implying that $y|\{x\}$. Thus $x|\{y\}$ and $\sqrt{xM} = \sqrt{yM}$. The argument is reversible.

3. Assume that there exists $z \in M \setminus M^\times$ with $z\|T$, $x \in T$, and $|T| \geq 3$. Writing $T = \{x, t_1, t_2, \cdots, t_k\}$, we see that $z|\{x, t_1t_2 \cdots t_k\}$, and thus, by hypothesis, $x|\{z\}$. Thus x is strongly homogeneous. The other implication is obvious.

4. Pick any nonunit z with $z|x$. Choose $y \in M \setminus M^{\times}$ with $y|\{z\}$, and assume that x is homogeneous. Then y divides a power of z, hence y divides a power of x and $y|\{x\}$. Therefore $x|\{y\}$ so $z|\{y\}$.

Now, assume that x is strongly homogeneous. Let $y\|S$ with $z \in S$. Set $S' = S \cup \{x\} \setminus \{z\}$. Now $y|S'$, and there is $T \subseteq S'$ with $y\|T$. If $x \notin T$, then $y|S \setminus \{z\}$, a contradiction. Hence $x|\{y\}$ and so $z|\{y\}$.

5. Let x be homogeneous. Pick $y \in M \setminus M^{\times}$ with $y\|S$ and $x \in S$. We observe that every irreducible dividing y must also divide a singleton subset of S, and in fact, there must be some irreducible divisor π of y such that $\pi|\{x\}$ (otherwise, if every irreducible divisor of y divides $S \setminus \{x\}$, then $y|S \setminus \{x\}$, a contradiction). As x is homogeneous, we have $x|\{\pi\}$, hence x divides a power of y and $x|\{y\}$. Therefore x is strongly homogeneous. The other implication is obvious. □

Lemma 3.9. *Let M be an monoid, let $x \in M \setminus M^{\times}$, and suppose that $x\|S$ and $x|T$. If there exists $s \in S$ that is strongly homogeneous, then there exists $t \in T$ and a subset S' of $S \setminus \{s\}$ such that $x\|S' \cup \{t\}$.*

Proof. Writing $S = \{s, s_1, s_2, \cdots, s_k\}$, we see that $x\|S$ implies that $s|\{x\}$. Since $x|T$, we see that $s|T$, and (since $\mathrm{div}(s) = 1$), $s|\{t\}$ for some $t \in T$. Pick $a \in \mathbb{N}$ and $r \in M$ such that $sr = t^a$, and pick $\alpha \in M$ and $b \in \mathbb{N}$ with,

$$x\alpha = (ss_1s_2\cdots s_k)^b. \text{ Then, } x\alpha r^b = t^{ab}(s_1s_2\cdots s_k)^b,$$

implying that $x|\{t, s_1, s_2, \cdots, s_k\}$. Thus, there exists $R \subseteq \{t, s_1, s_2, \cdots, s_k\}$ such that $x\|R$. We must have $t \in R$, otherwise $x|S \setminus \{s\}$, a contradiction. Therefore, $R = S' \cup \{t\}$ for some subset S' of S. □

Corollary 3.10. *Let M be a monoid and let $x \in M \setminus M^{\times}$. If $x\|S$, and if each element of S is strongly homogeneous, then div$(x) = |S|$.*

Proof. Suppose $x\|T$. We write $S = \{s_1, s_2, \cdots, s_k\}$ and $T = \{t_1, t_2, \cdots, t_m\}$. Suppose that $k \le m$. Repeatedly applying Lemma 3.9, we find $T' \subseteq T$ such that $|T'| \le |S|$ and $x\|T'$. Since $x\|T$, we must, in fact, have $T = T'$, and therefore $m \le k$, implying that $\mathrm{div}(x) = k$. □

We now give an example to show that "strongly homogeneous" cannot be replaced by "homogeneous" in Lemma 3.9 or Corollary 3.10.

Example 3.11. Consider the following multiplicative subsemigroups of $\mathbb{N}$:

$$A = [2, 3, 5, 7, 2\cdot 3\cdot 11, 5\cdot 7\cdot 13], \text{ and } B = 2\cdot 3\cdot 5\cdot 7\ \mathbb{N},$$

and let $M = A \cup B$. Note that M is atomic, so for computing diversity, we need only consider subsets of $\mathcal{A}(M)$.

We first show that $\mathrm{div}(2\cdot 3\cdot 11) = 1$. Let $2\cdot 3\cdot 11\|S$. Then, some $s \in S \cap \mathcal{A}(M)$ is a multiple (in $\mathbb{N}$) of 11. If $s = 2\cdot 3\cdot 11$, then $S = \{s\}$, otherwise, $s \in B$, and hence $s = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot y$ for some $y \in \mathbb{N}$. But then $S = \{s\}$ again, since

$$(2\cdot 3\cdot 11)(2\cdot 3\cdot 7\cdot 5\cdot 11\cdot 5\cdot 7\cdot y^2) = s^2,$$

and $2\cdot 3\cdot 11|s^2$. Therefore, $\text{div}(2\cdot 3\cdot 11) = 1$. A similar argument shows that $\text{div}(5\cdot 7\cdot 13) = 1$.

Further, we claim that $2\cdot 3\cdot 11$ is homogeneous. Let $x\|\{2\cdot 3\cdot 11\}$ for some $x \in M \setminus \{1\}$. Then $xr = (2\cdot 3\cdot 11)^k$ for some $r \in M$ and $k \in \mathbb{N}$. However $(2\cdot 3\cdot 11)^k$ has unique factorization in M, since only atoms $2, 3, 2\cdot 3\cdot 11$ divide as integers, but in fact we must have k copies of $2\cdot 3\cdot 11$ since that is the only source of rational prime 11. Hence $x = (2\cdot 3\cdot 11)^j$ for some $j \in \mathbb{N}$, $2\cdot 3\cdot 11|x$, and in particular $2\cdot 3\cdot 11|\{x\}$. Thus, $2\cdot 3\cdot 11$ is homogeneous, and, similarly, so is $5\cdot 7\cdot 13$.

We next observe that $\text{div}(2\cdot 3\cdot 5\cdot 7) \geq 4$, as $2\cdot 3\cdot 5\cdot 7\|\{2, 3, 5, 7\}$. However, we also have $2\cdot 3\cdot 5\cdot 7\|\{2\cdot 3\cdot 11, 5\cdot 7\cdot 13\}$. Thus, the conclusion of Corollary 3.10 does not hold for M. Further, $2\cdot 3\cdot 5\cdot 7$ divides none of $\{2, 2\cdot 3\cdot 11\}, \{3, 2\cdot 3\cdot 11\}, \{5, 2\cdot 3\cdot 11\}, \{7, 2\cdot 3\cdot 11\}$ and hence the conclusion of Lemma 3.9 does not hold for M.

Proposition 3.12. *Let S, T be subsets of M consisting of strongly homogeneous elements. Suppose that $x\|S$ and $x\|T$. Then $|S| = |T|$ and $\{\sqrt{sM} : s \in S\} = \{\sqrt{tM} : t \in T\}$.*

Proof. By Corollary 3.10, $|S| = \text{div}(x) = |T|$. Choose $s \in S$. We have $s|\{x\}$ (since $x\|S$ and s is strongly homogeneous) and $x|T$. As $\text{div}(s) = 1$, we have $s|\{t\}$ for some $t \in T$. Therefore, $\sqrt{tM} \subseteq \sqrt{sM}$, and by Theorem 3.8, $\sqrt{tM} = \sqrt{sM}$. □

Corollary 3.13. *Let M be atomic. Then the following are equivalent:*

1. *Every atom of M is homogeneous.*

2. *For every nonunit $x \in M$ and for sets $S = \{\pi_1, \pi_2, \cdots, \pi_n\}$ and $T = \{\xi_1, \xi_2, \cdots, \xi_m\}$ of pairwise nonassociate atoms of M, if $x\|S$ and $x\|T$ then $|S| = |T|$ and for each $1 \leq i \leq n$, there exists a permutation $\sigma \in S_n$ such that $\sqrt{\pi_i M} = \sqrt{\xi_{\sigma(i)} M}$.*

Proof. Suppose first that every atom of M is homogeneous. Then every atom of M is strongly homogeneous by Theorem 3.8.5. The conclusion follows from Proposition 3.12.

Suppose now that the second condition holds. Choose $\pi \in \mathcal{A}(M)$. The hypothesis implies that π is almost primary and thus $\text{div}(x) = 1$. Now choose $y \in M \setminus M^\times$ with $y|\{\pi\}$. Factor y into atoms as $y = \xi_1 \cdots \xi_k$. Since $y|\{\xi_1, \ldots, \xi_k\}$, by hypothesis there is some ξ_i with $\sqrt{\pi M} = \sqrt{\xi_i M}$ and hence $\sqrt{\pi M} \supseteq \sqrt{\xi_i M}$. By Theorem 3.8.1, $\pi|\{\xi_i\}$ and so $\pi|\{y\}$. Hence π is homogeneous. □

4 Factorization into almost primary elements

Proposition 4.1. *Let M be a monoid and let $x \in M \setminus M^\times$. Let $x = y_1 y_2 \cdots y_n$ for $y_i \in M \setminus M^\times$. Then x is homogeneous (respectively strongly homogeneous) if and only if each y_i is homogeneous (respectively strongly homogeneous) and the y_i are pairwise related (or equivalently, $\sqrt{y_i M} = \sqrt{y_j M}$ for all $1 \leq i \leq j \leq n$).*

Proof. If x is homogeneous (resp. strongly homogeneous), then each y_i is homogeneous (resp. strongly homogeneous) by Theorem 3.8.4. Also, for each $1 \leq i \leq n$, $xM \subseteq y_iM$, implying that $\sqrt{xM} \subseteq \sqrt{y_iM}$. By Theorem 3.8.1, $\sqrt{xM} = \sqrt{y_iM}$.

On the other hand, if each y_i is homogeneous and if $\sqrt{y_iM} = \sqrt{y_jM}$ for each $1 \leq i \leq j \leq n$, then

$$\sqrt{xM} = \sqrt{y_1 y_2 \cdots y_n M} = \sqrt{y_1 M} \cap \sqrt{y_2 M} \cap \cdots \cap \sqrt{y_n M} = \sqrt{y_1 M}.$$

So, $\sqrt{xM}$ is prime and maximal amongst radicals of principal ideals, whence x is homogeneous by Theorem 3.8.2.

Now, assume that each y_i is strongly homogeneous and pairwise related. Then, $\sqrt{xM} = \sqrt{y_iM}$ for each i. Letting $z \in M \setminus M^\times$ with $z \| S$ and $S = \{x, s_1, s_2, \cdots, s_k\}$, the fact that $\sqrt{xM} = \sqrt{y_1M}$ implies that $x|\{y_1\}$, whence $z|\{y_i, s_1, s_2, \cdots, s_k\}$. So, $y_i|\{z\}$ for each i, implying that $x|\{z\}$. Therefore x is strongly homogeneous. □

Definition 4.2. Let M be a monoid, let $x \in M$, and suppose that

$$x = q_1 q_2 \cdots q_n,$$

where each q_i is almost primary. If $\sqrt{q_iM}$ and $\sqrt{q_jM}$ are incomparable for all $i \neq j$, we say that the above factorization is a **reduced factorization** of x into almost primary elements.

Clearly, if a nonunit element x of a monoid M can be factored into almost primary elements, then we may find a reduced factorization of x into almost primary elements, merely by consolidating almost primary divisors of x whose radicals are comparable.

In Theorem 1.5 of [5], Halter-Koch showed that reduced factorizations into primary elements are unique up to associates. In other words, if

$$q_1 q_2 \cdots q_n = r_1 r_2 \cdots r_m$$

are reduced factorizations of some nonunit x into primary elements, then $n = m$ and there exists $\sigma \in S_n$ such that q_i is associate to $r_{\sigma(i)}$.

If we consider factorizations into almost primary elements, then we need not have uniqueness. For example, consider Example 3.6. For distinct rational primes $p, q \in \mathbb{N}$, there are two reduced factorizations of $pq = (p)(q) = (-p)(-q)$. However, reduced factorizations into almost primary elements are unique up to length and radicals, as shown in the following.

Proposition 4.3. *Let M be a monoid, let $x \in M$, and suppose that*

$$q_1 q_2 \cdots q_n = r_1 r_2 \cdots r_m$$

are two reduced factorizations of x into almost primary elements. Then $div(x) = n = m$ and there exists $\sigma \in S_n$ such that $\sqrt{q_iM} = \sqrt{r_{\sigma(i)}M}$.

Proof. As q_1 is almost primary, we have (without loss of generality) that $q_1|r_1^k$ for some $k \in \mathbb{N}$. Thus, $\sqrt{r_1M} \subseteq \sqrt{q_1M}$. However, r_1 is almost primary, whence $r_1|q_i^t$ for some i, $1 \leq i \leq n$. However, we then have $\sqrt{q_iM} \subseteq \sqrt{r_1M} \subseteq \sqrt{q_1M}$, forcing $i = 1$ (as our factorizations of x are reduced). Therefore $\sqrt{q_1M} = \sqrt{r_1M}$. Applying induction, we see that $m = n$ and that we may pair up the q's and r's by radicals.

To show that $\text{div}(x) = n$, we first note that $\text{div}(x) \leq \sum_{i=1}^{n} \text{div}(q_i) = n$. Also, $x|\{q_1, q_2, \cdots, q_n\}$, and if $x|\{q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n\}$ for some $1 \leq i \leq n$, then $q_i|q_j^t$ for some $t \in \mathbb{N}$ and $j \neq i$. However, this would imply that $\sqrt{q_jM} \subseteq \sqrt{q_iM}$, a contradiction. Therefore $x\|\{q_1, q_2, \cdots, q_n\}$ and $\text{div}(x) \geq n$. □

This next theorem, informally speaking, considers conditions that are as close as we can hope to $\text{div}(\cdot) : M \setminus M^{\times} \to \mathbb{N} \cup \{\infty\}$ being a homomorphism of semigroups.

Theorem 4.4. *Let M be a monoid. For $x \in M \setminus M^{\times}$, the following are equivalent:*

1. *If $div(x) \geq 2$, then there exist $y, z \in M \setminus M^{\times}$ with $x = yz$ and $div(x) = div(y) + div(z)$.*
2. *x can be written as a product of $div(x)$ almost primary elements.*
3. *There exists a reduced factorization of x into almost primary elements.*

Additionally, if M is atomic, then $div_a(M) = 1$ if and only if 1-3 hold for every $x \in M \setminus M^{\times}$.

Proof. ($1 \Rightarrow 2$): If $\text{div}(x) = 1$, there is nothing to prove. If $\text{div}(x) = n \geq 2$, we may repeatedly apply 1 to obtain $x = y_1y_2\cdots y_n$ with $\text{div}(y_i) = 1$. It then follows that each y_i is almost primary.

($2 \Rightarrow 3$): Write x as a product of $\text{div}(x)$ almost primary elements and, if necessary, group factors with comparable radicals together.

($3 \Rightarrow 1$): Suppose that $x = q_1q_2\cdots q_n$ is such a reduced factorization and that $n \geq 2$. By Proposition 4.3, $\text{div}(x) = n = \sum_{i=1}^{n} \text{div}(q_i)$.

Now suppose 1-3 hold for all $x \in M \setminus M^{\times}$. Let $x \in M$ with $\text{div}(x) \geq 2$. By condition 1, x is reducible, hence $\text{div}_{\text{a}}(M) = 1$. On the other hand, suppose $\text{div}_{\text{a}}(M) = 1$. Then every atom of M is almost primary. Hence every $x \in M \setminus M^{\times}$ can be factored into almost primary elements, and thus has a reduced factorization into almost primary elements. □

Theorem 4.5. *Let M be a monoid. Then, the following are equivalent.*

1. *Every nonunit element of M has a reduced factorization into primary elements (i.e. M is a WFM).*
2. *Given $x \in M \setminus M^{\times}$, there exist homogeneous $q_1, q_2, \cdots, q_n$ (each with distinct radicals) such that $x = q_1q_2\cdots q_n$, where this factorization into homogeneous elements is unique in the sense that if $x = q_1'q_2'\cdots q_n'$ is any*

reduced factorization of x into almost primary elements, then there exists $\sigma \in S_n$ such that q_i is associate to $q'_{\sigma(i)}$.

Proof. (1⇒ 2): This follows from Theorem 1.5 of [5].

(2⇒ 1): It suffices to show that every homogeneous element is primary. Let $q \in M$ be homogeneous, and suppose that $q|ab$ for some $a, b \in M$. We may assume that a and b are nonunits of M (otherwise $q|a$ or $q|b$). So, we have $qr = ab$ for some $r \in M$. If $r \in M^\times$, then we have reduced factorizations

$$a = a_1 a_2 \cdots a_m \text{ and } b = b_1 b_2 \cdots b_n$$

of a and b into almost primary elements. However, by hypothesis, either q is an associate of a_1 and $b \in M^\times$ or q is an associate of b_1 and $a \in M^\times$, a contradiction. Therefore, we have a reduced factorization $r_1 r_2 \cdots r_k$ of r. It follows that

$$q r_1 r_2 \cdots r_k = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n.$$

However, we cannot necessarily conclude that the factorizations above are reduced, but, no more than one of the r_i can share the same radical as q (and likewise for the a_j and b_l). If $q|a_j$ or $q|b_l$ for some j or l, then $q|a$ or $q|b$.

If (without loss of generality), $\sqrt{a_1 M} = \sqrt{b_1 M}$ and $q = a_1 b_1$, then $a_1|\{q\}$ and $b_1|\{q\}$, whence $\sqrt{a_1 M} = \sqrt{qM} = \sqrt{b_1 M}$. Therefore, $q|\{a_1\}$, $q|\{b_1\}$, and q divides both a power of a and a power of b.

Therefore, we may assume that (without loss of generality) $\sqrt{q_1 M} = \sqrt{r_1 M}$, $\sqrt{a_1 M} = \sqrt{b_1 M}$, and $q r_1 = a_1 b_1$. Thus, $\sqrt{qM} = \sqrt{a_1 M} = \sqrt{b_1 M}$, and q divides both a power of a and a power of b.

We conclude that q is primary, and therefore M is weakly factorial. □

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