Factorization Theory of Matrices

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University

November 18, 2010
Karl Franzens University Graz

$\underbrace{\texttt{http://www-rohan.sdsu.edu/\~{}vadim}}$`/matrices.pdf`

San Diego State University

Motivation

Factorization Theory $\bigcap$ Matrix Theory $=$?

Joint work with:

Donald Adams, Rene Ardila, David Hannasch, Audra Kosh, Hanah McCarthy, Ryan Rosenbaum

Nicholas Baeth

Motivation

Factorization Theory $\bigcap$ Matrix Theory $=$?

Joint work with:

Donald Adams, Rene Ardila, David Hannasch, Audra Kosh, Hanah McCarthy, Ryan Rosenbaum

Nicholas Baeth

Strategy

Choose a collection of square matrices:

- Interesting
- Multiplicatively closed and atomic

Study arithmetic properties:

- units
- atoms
- factorization of arbitrary elements
- factorization invariants

Strategy

Choose a collection of square matrices:

- Interesting
- Multiplicatively closed and atomic

Study arithmetic properties:

- units
- atoms
- factorization of arbitrary elements
- factorization invariants

Ideas from Factorization Theory

S multiplicatively closed set.

$u \in S$ is a unit if there is $v \in S$ and $uv = 1$. $S^\times$

$a \in S$ is an atom if $a \notin (S - S^\times)(S - S^\times)$. $\mathcal{A}(S)$

For $x \in S - S^\times$, set $\mathcal{L}(x) = \{t : x = a_1 \cdots a_t, a_i \in \mathcal{A}(S)\}$.

Set $L(x) = \max \mathcal{L}(x), l(x) = \min \mathcal{L}(x), \rho(x) = L(x)/l(x)$.

S is half-factorial if $l(x) = L(x)$; S is bifurcus if $l(x) \leq 2$.
(for all $x \in S - S^\times$)

Set $\Delta(x) = \{d : t_1, t_2 \in \mathcal{L}(x),$ consecutive, $t_2 - t_1 = d\}$.

Set $\rho(S) = \sup \rho(x), \Delta(S) = \bigcup \Delta(x)$. (over all $x \in S$)

others...?

Ideas from Factorization Theory

S multiplicatively closed set.

$u \in S$ is a unit if there is $v \in S$ and $uv = 1$. $S^\times$

$a \in S$ is an atom if $a \notin (S - S^\times)(S - S^\times)$. $\mathcal{A}(S)$

For $x \in S - S^\times$, set $\mathcal{L}(x) = \{t : x = a_1 \cdots a_t, a_i \in \mathcal{A}(S)\}$.

Set $L(x) = \max \mathcal{L}(x), l(x) = \min \mathcal{L}(x), \rho(x) = L(x)/l(x)$.

S is half-factorial if $l(x) = L(x)$; S is bifurcus if $l(x) \leq 2$.
(for all $x \in S - S^\times$)

Set $\Delta(x) = \{d : t_1, t_2 \in \mathcal{L}(x),$ consecutive, $t_2 - t_1 = d\}$.

Set $\rho(S) = \sup \rho(x), \Delta(S) = \bigcup \Delta(x)$. (over all $x \in S$)

others...?

Ideas from Factorization Theory

S multiplicatively closed set.

$u \in S$ is a unit if there is $v \in S$ and $uv = 1$. $S^\times$

$a \in S$ is an atom if $a \notin (S - S^\times)(S - S^\times)$. $\mathcal{A}(S)$

For $x \in S - S^\times$, set $\mathcal{L}(x) = \{t : x = a_1 \cdots a_t, a_i \in \mathcal{A}(S)\}$.

Set $L(x) = \max \mathcal{L}(x), l(x) = \min \mathcal{L}(x), \rho(x) = L(x)/l(x)$.

S is half-factorial if $l(x) = L(x)$; S is bifurcus if $l(x) \leq 2$.
(for all $x \in S - S^\times$)

Set $\Delta(x) = \{d : t_1, t_2 \in \mathcal{L}(x),$ consecutive, $t_2 - t_1 = d\}$.

Set $\rho(S) = \sup \rho(x), \Delta(S) = \bigcup \Delta(x)$. (over all $x \in S$)

others...?

Ideas from Factorization Theory

S multiplicatively closed set.

$u \in S$ is a unit if there is $v \in S$ and $uv = 1$. $S^\times$

$a \in S$ is an atom if $a \notin (S - S^\times)(S - S^\times)$. $\mathcal{A}(S)$

For $x \in S - S^\times$, set $\mathcal{L}(x) = \{t : x = a_1 \cdots a_t, a_i \in \mathcal{A}(S)\}$.

Set $L(x) = \max \mathcal{L}(x), l(x) = \min \mathcal{L}(x), \rho(x) = L(x)/l(x)$.

S is half-factorial if $l(x) = L(x)$; S is bifurcus if $l(x) \leq 2$.
(for all $x \in S - S^\times$)

Set $\Delta(x) = \{d : t_1, t_2 \in \mathcal{L}(x),$ consecutive, $t_2 - t_1 = d\}$.

Set $\rho(S) = \sup \rho(x)$, $\Delta(S) = \bigcup \Delta(x)$. (over all $x \in S$)

others...?

SAN DIEGO STATE UNIVERSITY

Ideas from Factorization Theory

S multiplicatively closed set.

$u \in S$ is a unit if there is $v \in S$ and $uv = 1$. $S^\times$

$a \in S$ is an atom if $a \notin (S - S^\times)(S - S^\times)$. $\mathcal{A}(S)$

For $x \in S - S^\times$, set $\mathcal{L}(x) = \{t : x = a_1 \cdots a_t, a_i \in \mathcal{A}(S)\}$.

Set $L(x) = \max \mathcal{L}(x), l(x) = \min \mathcal{L}(x), \rho(x) = L(x)/l(x)$.

S is half-factorial if $l(x) = L(x)$; S is bifurcus if $l(x) \leq 2$.
(for all $x \in S - S^\times$)

Set $\Delta(x) = \{d : t_1, t_2 \in \mathcal{L}(x),$ consecutive, $t_2 - t_1 = d\}$.

Set $\rho(S) = \sup \rho(x)$, $\Delta(S) = \bigcup \Delta(x)$. (over all $x \in S$)

others...?

Choosing S

Today, will consider only subsets of $M_n(\mathbb{Z})$, for $n \in \mathbb{N}$.

Natural restrictions:

- Replace $\mathbb{Z}$ with sub-semiring, e.g. $\mathbb{N}$, $\mathbb{N}_0$, $m\mathbb{Z}$, $m\mathbb{N}$
- Determinants in some multiplicative subsemigroup of $\mathbb{Z}$, e.g. $0, \mathbb{Z}^\times, \mathbb{N}, m\mathbb{Z}, \mathbb{Z}^\bullet$ (no zero divisors), composites
- Matrix structure

Choosing S

Today, will consider only subsets of $M_n(\mathbb{Z})$, for $n \in \mathbb{N}$.

Natural restrictions:

- Replace $\mathbb{Z}$ with sub-semiring, e.g. $\mathbb{N}$, $\mathbb{N}_0$, $m\mathbb{Z}$, $m\mathbb{N}$
- Determinants in some multiplicative subsemigroup of $\mathbb{Z}$, e.g. $0, \mathbb{Z}^\times, \mathbb{N}, m\mathbb{Z}, \mathbb{Z}^\bullet$ (no zero divisors), composites
- Matrix structure

Choosing S

Today, will consider only subsets of $M_n(\mathbb{Z})$, for $n \in \mathbb{N}$.

Natural restrictions:

- Replace $\mathbb{Z}$ with sub-semiring, e.g. $\mathbb{N}$, $\mathbb{N}_0$, $m\mathbb{Z}$, $m\mathbb{N}$
- Determinants in some multiplicative subsemigroup of $\mathbb{Z}$, e.g. $0, \mathbb{Z}^\times, \mathbb{N}, m\mathbb{Z}, \mathbb{Z}^\bullet$ (no zero divisors), composites
- Matrix structure

Choosing S

Today, will consider only subsets of $M_n(\mathbb{Z})$, for $n \in \mathbb{N}$.

Natural restrictions:

- Replace $\mathbb{Z}$ with sub-semiring, e.g. $\mathbb{N}$, $\mathbb{N}_0$, $m\mathbb{Z}$, $m\mathbb{N}$
- Determinants in some multiplicative subsemigroup of $\mathbb{Z}$, e.g. $0, \mathbb{Z}^\times, \mathbb{N}, m\mathbb{Z}, \mathbb{Z}^\bullet$ (no zero divisors), composites
- Matrix structure

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

SAN DIEGO STATE UNIVERSITY

Matrix Structure

$A = (a_{i,j})$ is:

- Triangular, if $a_{i,j} = 0$ for $i > j$.
- Unitriangular, if triangular and $a_{i,i} = 1$.
- Gaussian, if unitriangular and $\exists k$, $a_{i,j} = 0$ for $j \neq k, i$.
- Rank 1, if there is exactly one nonzero eigenvalue.
- Single-valued, if for some k, all $a_{i,j} = k$.
- Bistochastic, if all row and column sums are equal, i.e. $\sum_i a_{i,j} = \sum_j a_{i,j}$

Previous Work

[1966 Jacobson Wisner] $S = M_2(\mathbb{N}_0)$ with determinant=1. Then $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the only atoms, and every $x \in S$ has unique factorization.

Also, $S = M_2(\mathbb{Z})$ with determinant=1 has no atoms.

[1985 Chuan Chuan] $S = M_2(\mathbb{N})$ with determinant=1. $x \in S$ is an atom exactly when at least one of its entries is 1. Some atoms have two 1's in a row A_r, some atoms have two 1's in a column A_c. Canonical factorizations were found maximizing the number of atoms from A_r, A_c, which is the same. However, S is not half-factorial.

Conjectures: $\rho(S) \leq 2, \Delta(S) = 1$.

San Diego State University

Previous Work

[1966 Jacobson Wisner] $S = M_2(\mathbb{N}_0)$ with determinant=1. Then $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the only atoms, and every $x \in S$ has unique factorization.

Also, $S = M_2(\mathbb{Z})$ with determinant=1 has no atoms.

[1985 Chuan Chuan] $S = M_2(\mathbb{N})$ with determinant=1. $x \in S$ is an atom exactly when at least one of its entries is 1. Some atoms have two 1's in a row A_r, some atoms have two 1's in a column A_c. Canonical factorizations were found maximizing the number of atoms from A_r, A_c, which is the same. However, S is not half-factorial.

Conjectures: $\rho(S) \leq 2, \Delta(S) = 1$.

Previous Work

[1966 Jacobson Wisner] $S = M_2(\mathbb{N}_0)$ with determinant=1. Then $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the only atoms, and every $x \in S$ has unique factorization.

Also, $S = M_2(\mathbb{Z})$ with determinant=1 has no atoms.

[1985 Chuan Chuan] $S = M_2(\mathbb{N})$ with determinant=1. $x \in S$ is an atom exactly when at least one of its entries is 1. Some atoms have two 1's in a row A_r, some atoms have two 1's in a column A_c. Canonical factorizations were found maximizing the number of atoms from A_r, A_c, which is the same. However, S is not half-factorial.

Conjectures: $\rho(S) \leq 2, \Delta(S) = 1$.

Previous Work

[1966 Jacobson Wisner] $S = M_2(\mathbb{N}_0)$ with determinant=1. Then $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the only atoms, and every $x \in S$ has unique factorization.

Also, $S = M_2(\mathbb{Z})$ with determinant=1 has no atoms.

[1985 Chuan Chuan] $S = M_2(\mathbb{N})$ with determinant=1. $x \in S$ is an atom exactly when at least one of its entries is 1. Some atoms have two 1's in a row A_r, some atoms have two 1's in a column A_c. Canonical factorizations were found maximizing the number of atoms from A_r, A_c, which is the same. However, S is not half-factorial.

Conjectures: $\rho(S) \leq 2, \Delta(S) = 1$.

Previous Work

[1966 Jacobson Wisner] $S = M_2(\mathbb{N}_0)$ with determinant=1. Then $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the only atoms, and every $x \in S$ has unique factorization.

Also, $S = M_2(\mathbb{Z})$ with determinant=1 has no atoms.

[1985 Chuan Chuan] $S = M_2(\mathbb{N})$ with determinant=1. $x \in S$ is an atom exactly when at least one of its entries is 1. Some atoms have two 1's in a row A_r, some atoms have two 1's in a column A_c. Canonical factorizations were found maximizing the number of atoms from A_r, A_c, which is the same. However, S is not half-factorial.

Conjectures: $\rho(S) \leq 2, \Delta(S) = 1$.

Transfer Homomorphisms

S as before, T a semigroup, $\eta : S \to T$ a homomorphism.

Certainly $\eta(s_1 s_2) = \eta(s_1)\eta(s_2)$ because η is a homomorphism; η is transfer if the "converse" holds, i.e.:

If $\eta(s) = t_1 t_2$ with $t_1, t_2 \in T - T^\times$, then we can find $s_1, s_2 \in S - S^\times$ with $s = s_1 s_2$ and $t_1 = \eta(s_1), t_2 = \eta(s_2)$.

Factorization in S follows factorization in T.

Transfer Homomorphisms

S as before, T a semigroup, $\eta : S \to T$ a homomorphism.

Certainly $\eta(s_1 s_2) = \eta(s_1)\eta(s_2)$ because η is a homomorphism; η is transfer if the "converse" holds, i.e.:

If $\eta(s) = t_1 t_2$ with $t_1, t_2 \in T - T^\times$, then we can find $s_1, s_2 \in S - S^\times$ with $s = s_1 s_2$ and $t_1 = \eta(s_1), t_2 = \eta(s_2)$.

Factorization in S follows factorization in T.

Transfer Examples

Determinant is multiplicative, a homomorhism from S to $\mathbb{Z}$.

Therefore: $|u| = 1$ for all units $u \in S$. (don't need transfer)

Thm. det: $M_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

atoms, $L(A) = l(A)$, half-factorial

Thm. det: $T_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

Transfer Examples

Determinant is multiplicative, a homomorhism from S to $\mathbb{Z}$.

Therefore: $|u| = 1$ for all units $u \in S$. (don't need transfer)

Thm. det: $M_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

atoms, $L(A) = l(A)$, half-factorial

Thm. det: $T_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

Transfer Examples

Determinant is multiplicative, a homomorhism from S to $\mathbb{Z}$.
Therefore: $|u| = 1$ for all units $u \in S$. (don't need transfer)

Thm. det: $M_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

atoms, $L(A) = l(A)$, half-factorial

Thm. det: $T_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

Transfer Examples

Determinant is multiplicative, a homomorhism from S to $\mathbb{Z}$.
Therefore: $|u| = 1$ for all units $u \in S$. (don't need transfer)

Thm. det: $M_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.
atoms, $L(A) = l(A)$, half-factorial

Thm. det: $T_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

Transfer Examples

Determinant is multiplicative, a homomorhism from S to $\mathbb{Z}$.
Therefore: $|u| = 1$ for all units $u \in S$. (don't need transfer)

Thm. det: $M_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.
atoms, $L(A) = l(A)$, half-factorial

Thm. det: $T_n(\mathbb{Z})^\bullet \to \mathbb{Z}^\bullet$ is a transfer homomorphism.

Restricting Determinant

Let S be those elements of $\{M_n(\mathbb{Z})^\bullet, T_n(\mathbb{Z})^\bullet\}$ whose determinant is in a multiplicatively closed subset $T \subseteq \mathbb{Z}^\bullet$. Then det: $S \rightarrow T$ is transfer.

- $T = \{x : x > 1\}$
- $T = \{x : x \text{ is composite }\}$

Let $r : \mathbb{Z}^\bullet \rightarrow \mathbb{N}_0$ counts prime factors (with multiplicity).

Thm. $r : T \rightarrow < 2, 3 >$ is a transfer homomorphism, where $< 2, 3 >$ is a subsemigroup of $(\mathbb{N}_0, +)$

atoms, $L(A) = \lfloor \frac{r(|A|)}{2} \rfloor, l(a) = \lceil \frac{r(|A|)}{3} \rceil$,
$\rho(S) = 1.5, \Delta(S) = \{1\}$.

SAN DIEGO STATE UNIVERSITY

Restricting Determinant

Let S be those elements of $\{M_n(\mathbb{Z})^\bullet, T_n(\mathbb{Z})^\bullet\}$ whose determinant is in a multiplicatively closed subset $T \subseteq \mathbb{Z}^\bullet$. Then det: $S \to T$ is transfer.

- $T = \{x : x > 1\}$

- $T = \{x : x \text{ is composite }\}$

Let $r : \mathbb{Z}^\bullet \to \mathbb{N}_0$ counts prime factors (with multiplicity).

Thm. $r : T \to < 2, 3 >$ is a transfer homomorphism, where $< 2, 3 >$ is a subsemigroup of $(\mathbb{N}_0, +)$

atoms, $L(A) = \lfloor \frac{r(|A|)}{2} \rfloor, l(a) = \lceil \frac{r(|A|)}{3} \rceil$,
$\rho(S) = 1.5, \Delta(S) = \{1\}$.

Restricting Determinant

Let S be those elements of $\{M_n(\mathbb{Z})^\bullet, T_n(\mathbb{Z})^\bullet\}$ whose determinant is in a multiplicatively closed subset $T \subseteq \mathbb{Z}^\bullet$. Then det: $S \to T$ is transfer.

- $T = \{x : x > 1\}$
- $T = \{x : x \text{ is composite }\}$

Let $r : \mathbb{Z}^\bullet \to \mathbb{N}_0$ counts prime factors (with multiplicity).

Thm. $r : T \to < 2, 3 >$ is a transfer homomorphism, where $< 2, 3 >$ is a subsemigroup of $(\mathbb{N}_0, +)$

atoms, $L(A) = \lfloor \frac{r(|A|)}{2} \rfloor, l(a) = \lceil \frac{r(|A|)}{3} \rceil$,
$\rho(S) = 1.5, \Delta(S) = \{1\}$.

San Diego State University

Restricting Determinant

Let S be those elements of $\{M_n(\mathbb{Z})^\bullet, T_n(\mathbb{Z})^\bullet\}$ whose determinant is in a multiplicatively closed subset $T \subseteq \mathbb{Z}^\bullet$. Then det: $S \to T$ is transfer.

- $T = \{x : x > 1\}$
- $T = \{x : x \text{ is composite }\}$

Let $r : \mathbb{Z}^\bullet \to \mathbb{N}_0$ counts prime factors (with multiplicity).

Thm. $r : T \to < 2, 3 >$ is a transfer homomorphism, where $< 2, 3 >$ is a subsemigroup of $(\mathbb{N}_0, +)$

atoms, $L(A) = \lfloor \frac{r(|A|)}{2} \rfloor, l(a) = \lceil \frac{r(|A|)}{3} \rceil$,
$\rho(S) = 1.5, \Delta(S) = \{1\}$.

San Diego State University

Restricting Determinant

Let S be those elements of $\{M_n(\mathbb{Z})^\bullet, T_n(\mathbb{Z})^\bullet\}$ whose determinant is in a multiplicatively closed subset $T \subseteq \mathbb{Z}^\bullet$. Then det: $S \to T$ is transfer.

- $T = \{x : x > 1\}$

- $T = \{x : x \text{ is composite }\}$

Let $r : \mathbb{Z}^\bullet \to \mathbb{N}_0$ counts prime factors (with multiplicity).

Thm. $r : T \to < 2, 3 >$ is a transfer homomorphism, where $< 2, 3 >$ is a subsemigroup of $(\mathbb{N}_0, +)$

atoms, $L(A) = \lfloor \frac{r(|A|)}{2} \rfloor, l(a) = \lceil \frac{r(|A|)}{3} \rceil$,
$\rho(S) = 1.5, \Delta(S) = \{1\}$.

San Diego State University

Determinant in $m\mathbb{Z}$

Fix $m > 0$ and let $T = m\mathbb{Z}$.

For $x, y \in \mathbb{Z}^\bullet$, let $\nu_y(x) = \max\{s : y^s | x\}$.

Thm. Let $A \in S$. Then:

- $L(A) = \nu_m(|A|)$
- If m is not a prime power, then S is bifurcus.
- If $m = p^k$ then $l(A) = \lceil \frac{\nu_p(|A|)+2k-2}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

San Diego State University

Determinant in $m\mathbb{Z}$

Fix $m > 0$ and let $T = m\mathbb{Z}$.

For $x, y \in \mathbb{Z}^\bullet$, let $\nu_y(x) = \max\{s : y^s | x\}$.

Thm. Let $A \in S$. Then:

- $L(A) = \nu_m(|A|)$
- If m is not a prime power, then S is bifurcus.
- If $m = p^k$ then $l(A) = \lceil \frac{\nu_p(|A|)+2k-2}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

Determinant in $m\mathbb{Z}$

Fix $m > 0$ and let $T = m\mathbb{Z}$.

For $x, y \in \mathbb{Z}^\bullet$, let $\nu_y(x) = \max\{s : y^s | x\}$.

Thm. Let $A \in S$. Then:

- $L(A) = \nu_m(|A|)$
- If m is not a prime power, then S is bifurcus.
- If $m = p^k$ then $l(A) = \lceil \frac{\nu_p(|A|)+2k-2}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

Determinant in $m\mathbb{Z}$

Fix $m > 0$ and let $T = m\mathbb{Z}$.

For $x, y \in \mathbb{Z}^{\bullet}$, let $\nu_y(x) = \max\{s : y^s | x\}$.

Thm. Let $A \in S$. Then:

- $L(A) = \nu_m(|A|)$
- If m is not a prime power, then S is bifurcus.
- If $m = p^k$ then $l(A) = \lceil \frac{\nu_p(|A|)+2k-2}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

San Diego State University

Determinant in $m\mathbb{Z}$

Fix $m > 0$ and let $T = m\mathbb{Z}$.

For $x, y \in \mathbb{Z}^{\bullet}$, let $\nu_y(x) = \max\{s : y^s | x\}$.

Thm. Let $A \in S$. Then:

- $L(A) = \nu_m(|A|)$
- If m is not a prime power, then S is bifurcus.
- If $m = p^k$ then $l(A) = \lceil \frac{\nu_p(|A|)+2k-2}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^\bullet$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^{\bullet}$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

San Diego State University

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^\bullet$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

San Diego State University

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^{\bullet}$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

SAN DIEGO STATE UNIVERSITY

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^\bullet$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

Single-valued matrices

Let $S = \{A \in M_n(\mathbb{Z}) : \exists a \in \mathbb{Z}, \forall i, j, a_{i,j} = a\}$. Write $A = [a]$.

Properties: $[a] + [b] = [a + b], [a][b] = [nab]$, semiring.

Note that $\phi : S \to n\mathbb{Z}$ via $\phi([a]) = na$ is a transfer homomorphism.

Thm. Let $[a] \in S^\bullet$. Then:

- $L([a]) = \nu_n(a) + 1$
- If n is not a prime power, then S is bifurcus.
- If $n = p^k$, then $l([a]) = \lceil \frac{\nu_p(a)+k}{2k-1} \rceil$ and $\rho(S) = \frac{2k-1}{k}$
- $\Delta(S) \subseteq \{1\}$

San Diego State University

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.

atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.

atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.

no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Gaussian Matrices

Let G_n^k be those unitriangular matrices with all nonzero offdiagonal entries located in column k. Commutative(!). Let $\phi : G_n^k \to \mathbb{Z}^{n-k}$ take the entries below the diagonal in column k.

Thm. $\phi : G_n^k(\mathbb{N}_0) \to \mathbb{N}_0^{n-k}$ is a transfer homomorphism.
atoms, factorial

Thm. $\phi : G_n^k(\mathbb{N}) \to \mathbb{N}^{n-k}$ is a transfer homomorphism.
atoms, factorial/bifurcus

Thm. $\phi : G_n^k(\mathbb{Z}) \to \mathbb{Z}^{n-k}$ is a transfer homomorphism.
no atoms

Preliminaries

Set $R_n = \{A \in M_n, \text{ rank of } A = 1\}$.

Matrix theory: $A \in R_n(\mathbb{Z})$ if and only if there are vectors $u, v \in \mathbb{Z}^n$ with $A = uv^T$.

Note that if $u, v \in \mathbb{N}^n$, then $A \in M_n(\mathbb{N}) = M_n(\mathbb{N})^\bullet$.

Thm. Let $A = uv^T \in R_n(\mathbb{Z}^\bullet)$. Then gcd($A$)=gcd($u$)gcd($v$)

Preliminaries

Set $R_n = \{A \in M_n, \text{ rank of } A = 1\}$.

Matrix theory: $A \in R_n(\mathbb{Z})$ if and only if there are vectors $u, v \in \mathbb{Z}^n$ with $A = uv^T$.

Note that if $u, v \in \mathbb{N}^n$, then $A \in M_n(\mathbb{N}) = M_n(\mathbb{N})^\bullet$.

Thm. Let $A = uv^T \in R_n(\mathbb{Z}^\bullet)$. Then gcd($A$)=gcd($u$)gcd($v$)

Preliminaries

Set $R_n = \{A \in M_n, \text{ rank of } A = 1\}$.

Matrix theory: $A \in R_n(\mathbb{Z})$ if and only if there are vectors $u, v \in \mathbb{Z}^n$ with $A = uv^T$.

Note that if $u, v \in \mathbb{N}^n$, then $A \in M_n(\mathbb{N}) = M_n(\mathbb{N})^\bullet$.

Thm. Let $A = uv^T \in R_n(\mathbb{Z}^\bullet)$. Then gcd($A$)=gcd($u$)gcd($v$)

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1 (v_1^T u_2) v_2^T = (v_1^T u_2) u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1) 1 \cdots 1], y = [1 1 \cdots 1]^T$. Write $A = \gcd(A) B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y) u v^T = u (x^T y) v^T = (u x^T)(y v^T)$.
$\gcd(u x^T) = gcd(u) gcd(x) = 1$.
$\gcd(y v^T) = gcd(y) gcd(v) = 1$. So $u x^T, y v^T$ are atoms.

San Diego State University

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1 (v_1^T u_2) v_2^T = (v_1^T u_2) u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1) 1 \cdots 1], y = [11 \cdots 1]^T$. Write $A = \gcd(A) B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y) uv^T = u(x^T y) v^T = (ux^T)(yv^T)$.
$\gcd(ux^T) = gcd(u) gcd(x) = 1$.
$\gcd(yv^T) = gcd(y) gcd(v) = 1$. So ux^T, yv^T are atoms.

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1 (v_1^T u_2) v_2^T = (v_1^T u_2) u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1) 1 \cdots 1], y = [11 \cdots 1]^T$. Write $A = \gcd(A) B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y) uv^T = u(x^T y) v^T = (ux^T)(yv^T)$.
$\gcd(ux^T) = gcd(u) gcd(x) = 1$.
$\gcd(yv^T) = gcd(y) gcd(v) = 1$. So ux^T, yv^T are atoms.

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1 (v_1^T u_2) v_2^T = (v_1^T u_2) u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for
$x^T = [(\gcd(A) - n + 1) 1 \cdots 1], y = [11 \cdots 1]^T$. Write
$A = \gcd(A) B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y) u v^T = u (x^T y) v^T = (u x^T)(y v^T)$.
$\gcd(u x^T) = gcd(u) gcd(x) = 1$.
$\gcd(y v^T) = gcd(y) gcd(v) = 1$. So ux^T, yv^T are atoms.

San Diego State University

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1(v_1^T u_2) v_2^T = (v_1^T u_2) u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1) 1 \cdots 1], y = [11 \cdots 1]^T$. Write $A =$gcd$(A)B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y) uv^T = u(x^T y) v^T = (ux^T)(yv^T)$.
$\gcd(ux^T) = gcd(u) gcd(x) = 1$.
$\gcd(yv^T) = gcd(y) gcd(v) = 1$. So ux^T, yv^T are atoms.

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1(v_1^T u_2)v_2^T = (v_1^T u_2)u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1)1 \cdots 1], y = [11 \cdots 1]^T$. Write $A =\gcd(A)B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y)uv^T = u(x^T y)v^T = (ux^T)(yv^T)$.
$\gcd(ux^T) = gcd(u)gcd(x) = 1$.
$\gcd(yv^T) = gcd(y)gcd(v) = 1$. So ux^T, yv^T are atoms.

San Diego State University

The only proof today

Thm. $A \in R$ is an atom iff $\gcd(A) < n$. Also R is bifurcus.

Proof. Suppose
$A = A_1 A_2 = (u_1 v_1^T)(u_2 v_2^T) = u_1(v_1^T u_2)v_2^T = (v_1^T u_2)u_1 v_2^T$.
Hence $\gcd(A) \geq v_1^T u_2 \geq n$.

Now suppose $\gcd(A) \geq n$; we write $\gcd(A) = x^T y$ for $x^T = [(\gcd(A) - n + 1)1 \cdots 1], y = [11 \cdots 1]^T$. Write $A =$gcd$(A)B$ for some $B = uv^T$ with $\gcd(B) = 1$.
$A = (x^T y)uv^T = u(x^T y)v^T = (ux^T)(yv^T)$.
gcd$(ux^T) = gcd(u)gcd(x) = 1$.
gcd$(yv^T) = gcd(y)gcd(v) = 1$. So ux^T, yv^T are atoms.

L(A) in $R_n(\mathbb{N})$

Fix $A \in R_n(\mathbb{N})$. Factor $\gcd(A) = m_0 m_1 \cdots m_t$ with $1 \leq m_0 < n \leq m_i$ $(i \geq 1)$, to maximize t.

Note: If $n = 2$, then $t = r(gcd(A))$; otherwise $t \leq r(gcd(A))$.

Thm. $L(A) = t + 1$, where t maximal as above.

Thm. Finding t is an NP-complete problem.

L(A) in $R_n(\mathbb{N})$

Fix $A \in R_n(\mathbb{N})$. Factor $\gcd(A) = m_0 m_1 \cdots m_t$ with $1 \leq m_0 < n \leq m_i$ $(i \geq 1)$, to maximize t.

Note: If $n = 2$, then $t = r(gcd(A))$; otherwise $t \leq r(gcd(A))$.

Thm. $L(A) = t + 1$, where t maximal as above.

Thm. Finding t is an NP-complete problem.

L(A) in $R_n(\mathbb{N})$

Fix $A \in R_n(\mathbb{N})$. Factor $\gcd(A) = m_0 m_1 \cdots m_t$ with $1 \leq m_0 < n \leq m_i$ $(i \geq 1)$, to maximize t.

Note: If $n = 2$, then $t = r(gcd(A))$; otherwise $t \leq r(gcd(A))$.

Thm. $L(A) = t + 1$, where t maximal as above.

Thm. Finding t is an NP-complete problem.

San Diego State University

L(A) in $R_n(\mathbb{N})$

Fix $A \in R_n(\mathbb{N})$. Factor $\gcd(A) = m_0 m_1 \cdots m_t$ with $1 \leq m_0 < n \leq m_i$ $(i \geq 1)$, to maximize t.

Note: If $n = 2$, then $t = r(gcd(A))$; otherwise $t \leq r(gcd(A))$.

Thm. $L(A) = t + 1$, where t maximal as above.

Thm. Finding t is an NP-complete problem.

$R_n(m\mathbb{N})$

Fix $m \in \mathbb{N}$, consider $R_n(m\mathbb{N})$.

Thm. Let $A \in R_n(m\mathbb{N})$. A is an atom if and only if:

- $m^2 \nmid gcd(A)$, or
- $A = m^2 B$, and B is an atom of $R_n(\mathbb{N})$

Cor. $R_n(m\mathbb{N})$ is bifurcus.

$R_n(m\mathbb{N})$

Fix $m \in \mathbb{N}$, consider $R_n(m\mathbb{N})$.

Thm. Let $A \in R_n(m\mathbb{N})$. A is an atom if and only if:

- $m^2 \nmid gcd(A)$, or
- $A = m^2 B$, and B is an atom of $R_n(\mathbb{N})$

Cor. $R_n(m\mathbb{N})$ is bifurcus.

$R_n(m\mathbb{N})$

Fix $m \in \mathbb{N}$, consider $R_n(m\mathbb{N})$.

Thm. Let $A \in R_n(m\mathbb{N})$. A is an atom if and only if:

- $m^2 \nmid gcd(A)$, or
- $A = m^2 B$, and B is an atom of $R_n(\mathbb{N})$

Cor. $R_n(m\mathbb{N})$ is bifurcus.

$T_n(m\mathbb{Z})$

Set $S = T_n(m\mathbb{Z})$, with $n, m \geq 2$.

Thm. Let $A \in S$. Then $L(A) = \nu_m(gcd(A))$. Also, S is bifurcus.

Contrast with:

- For $S = T_n(\mathbb{Z}), M_n(\mathbb{Z})$ with determinant in $m\mathbb{Z}$, $L(A) = \nu_m(|A|)$ and $m = p^k$ determines ρ.
- For $S = \{[a] : a \in \mathbb{Z}\}$, $L(A) = \nu_n(a) + 1$ and $n = p^k$ determines ρ.

$T_n(m\mathbb{Z})$

Set $S = T_n(m\mathbb{Z})$, with $n, m \geq 2$.

Thm. Let $A \in S$. Then $L(A) = \nu_m(gcd(A))$. Also, S is bifurcus.

Contrast with:

- For $S = T_n(\mathbb{Z}), M_n(\mathbb{Z})$ with determinant in $m\mathbb{Z}$, $L(A) = \nu_m(|A|)$ and $m = p^k$ determines ρ.
- For $S = \{[a] : a \in \mathbb{Z}\}$, $L(A) = \nu_n(a) + 1$ and $n = p^k$ determines ρ.

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

Unitriangular Matrices

Set S to be the set of n-triangular matrices with 1 along the diagonal.

For $A \in S(\mathbb{N}_0)$, set $\sigma(A) = \sum_{i<j} a_{i,j}$.

Thm. For $A \in S(\mathbb{N}_0)$, $L(A) = \sigma(A)$.

Thm. If $n = 2$ then $S(\mathbb{N}_0)$ is factorial; if $n \geq 3$ then $\rho(S(\mathbb{N}_0)) = \infty$.

Open: $l(A), \Delta(A)$

Thm. Let $n \geq 4$. Then $S(\mathbb{N})$ is bifurcus.

Note: $n = 2$ Gaussian, but $n = 3$ is open

$T_2(\mathbb{N})$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}$.

Thm. Let $A \in S$. Then $L(A) = b$. Also, $\rho(S) = \infty$ and $p - 1 \in \Delta(S)$ for every rational prime p.

Conjecture: $\Delta(S) = 2\mathbb{N} \cup \{1\}$.

Open: $l(A)$

$T_2(\mathbb{N})$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}$.

Thm. Let $A \in S$. Then $L(A) = b$. Also, $\rho(S) = \infty$ and $p - 1 \in \Delta(S)$ for every rational prime p.

Conjecture: $\Delta(S) = 2\mathbb{N} \cup \{1\}$.

Open: $l(A)$

$T_2(\mathbb{N})$

$S = \left\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right\}$, with $a, b, c \in \mathbb{N}$.

Thm. Let $A \in S$. Then $L(A) = b$. Also, $\rho(S) = \infty$ and $p - 1 \in \Delta(S)$ for every rational prime p.

Conjecture: $\Delta(S) = 2\mathbb{N} \cup \{1\}$.

Open: $l(A)$

$T_2(\mathbb{N}_0)^\bullet$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}_0, ac \neq 0$.

Thm. $\mathcal{A}(S)$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (p rational prime).

Recall $r(x)$ counts prime factors of x by multiplicity.

Thm. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$. Then $L(A) = r(ac) + b$.
If $b = 0$ then $l(A) = r(ac)$. If $b | ac$ then $l(A) = r(ac) + 1$.
Also, $\rho(S) = \infty$.

Open: $l(A), \Delta(A)$

$T_2(\mathbb{N}_0)^\bullet$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}_0, ac \neq 0$.

Thm. $\mathcal{A}(S)$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (p rational prime).

Recall $r(x)$ counts prime factors of x by multiplicity.

Thm. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$. Then $L(A) = r(ac) + b$.
If $b = 0$ then $l(A) = r(ac)$. If $b|ac$ then $l(A) = r(ac) + 1$.
Also, $\rho(S) = \infty$.

Open: $l(A), \Delta(A)$

$T_2(\mathbb{N}_0)^\bullet$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}_0, ac \neq 0$.

Thm. $\mathcal{A}(S)$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (p rational prime).

Recall $r(x)$ counts prime factors of x by multiplicity.

Thm. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$. Then $L(A) = r(ac) + b$.
If $b = 0$ then $l(A) = r(ac)$. If $b|ac$ then $l(A) = r(ac) + 1$.
Also, $\rho(S) = \infty$.

Open: $l(A), \Delta(A)$

$T_2(\mathbb{N}_0)^\bullet$

$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$, with $a, b, c \in \mathbb{N}_0, ac \neq 0$.

Thm. $\mathcal{A}(S)$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (p rational prime).

Recall $r(x)$ counts prime factors of x by multiplicity.
Thm. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$. Then $L(A) = r(ac) + b$.
If $b = 0$ then $l(A) = r(ac)$. If $b | ac$ then $l(A) = r(ac) + 1$.
Also, $\rho(S) = \infty$.

Open: $l(A), \Delta(A)$

Bistochastic 2-matrices

$S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

Bistochastic 2-matrices

$S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

Bistochastic 2-matrices

$S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

Bistochastic 2-matrices

$S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

Bistochastic 2-matrices

$S = \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

Bistochastic 2-matrices

$S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$, with $a, b \in \mathbb{N}$.

Suppose $|A|$ is odd. Then $a \equiv b \pmod 2$. Set $u = \frac{a+b}{2}, v = \frac{a-b}{2}$. Note that $u, v \in \mathbb{Z}$ with $u \geq |v| + 2$ and $u \equiv v \pmod 2$. Write $A = [u, v]$.

Thm. $[u, v][x, y] = [ux, vy]$, $|A| = uv$. Let $d \in \mathbb{Z}$ be odd. If d is a square then there are no atoms with determinant d; otherwise there is exactly one atom with determinant d.

Preliminary Problem: Find atoms with determinant d.

Conjecture: If d is a square, then one if $16|d$, none if $16 \nmid d$. If d is a nonsquare, then two if $16|d$, one if $16 \nmid d$.

Open: almost everything

San Diego State University