Determining Integer-Valued Polynomials From Their Image

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University
(this year at Karl Franzens University Graz)

November 30, 2010
Third International Meeting on Integer-Valued Polynomials and Problems in Commutative Algebra

http://www-rohan.sdsu.edu/~vadim/marseille.pdf

SAN DIEGO STATE UNIVERSITY

Problem

Joint work with Scott Chapman, preprint available

$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z})\}$

Given $f, g \in \mathrm{Int}(\mathbb{Z})$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Given $f, g \in \mathbb{R}[x]$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Problem

Joint work with Scott Chapman, preprint available

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z})\}$$

Given $f, g \in \mathrm{Int}(\mathbb{Z})$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Given $f, g \in \mathbb{R}[x]$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Problem

Joint work with Scott Chapman, preprint available

$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z})\}$

Given $f, g \in \mathrm{Int}(\mathbb{Z})$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Given $f, g \in \mathbb{R}[x]$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Problem

Joint work with Scott Chapman, preprint available

$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z})\}$

Given $f, g \in \mathrm{Int}(\mathbb{Z})$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

Given $f, g \in \mathbb{R}[x]$ with $f(\mathbb{Z}) = g(\mathbb{Z})$, how are f, g related?

An Equivalence Relation

We say $f \sim g$ if for some $n \in \mathbb{Z}$, $f(x) = g(x - n)$ or $f(x) = g(-x - n)$.

Note 1: $\sim$ is an equivalence relation.

Note 2: If $f \sim g$ then $f(\mathbb{Z}) = g(\mathbb{Z})$. Converse?

Note 3: If $f \in \mathrm{Int}(\mathbb{Z})$ and $f \sim g$, then $g \in \mathrm{Int}(\mathbb{Z})$.

An Equivalence Relation

We say $f \sim g$ if for some $n \in \mathbb{Z}$, $f(x) = g(x - n)$ or $f(x) = g(-x - n)$.

Note 1: $\sim$ is an equivalence relation.

Note 2: If $f \sim g$ then $f(\mathbb{Z}) = g(\mathbb{Z})$. Converse?

Note 3: If $f \in \mathrm{Int}(\mathbb{Z})$ and $f \sim g$, then $g \in \mathrm{Int}(\mathbb{Z})$.

An Equivalence Relation

We say $f \sim g$ if for some $n \in \mathbb{Z}$, $f(x) = g(x - n)$ or $f(x) = g(-x - n)$.

Note 1: $\sim$ is an equivalence relation.

Note 2: If $f \sim g$ then $f(\mathbb{Z}) = g(\mathbb{Z})$. Converse?

Note 3: If $f \in \mathrm{Int}(\mathbb{Z})$ and $f \sim g$, then $g \in \mathrm{Int}(\mathbb{Z})$.

An Equivalence Relation

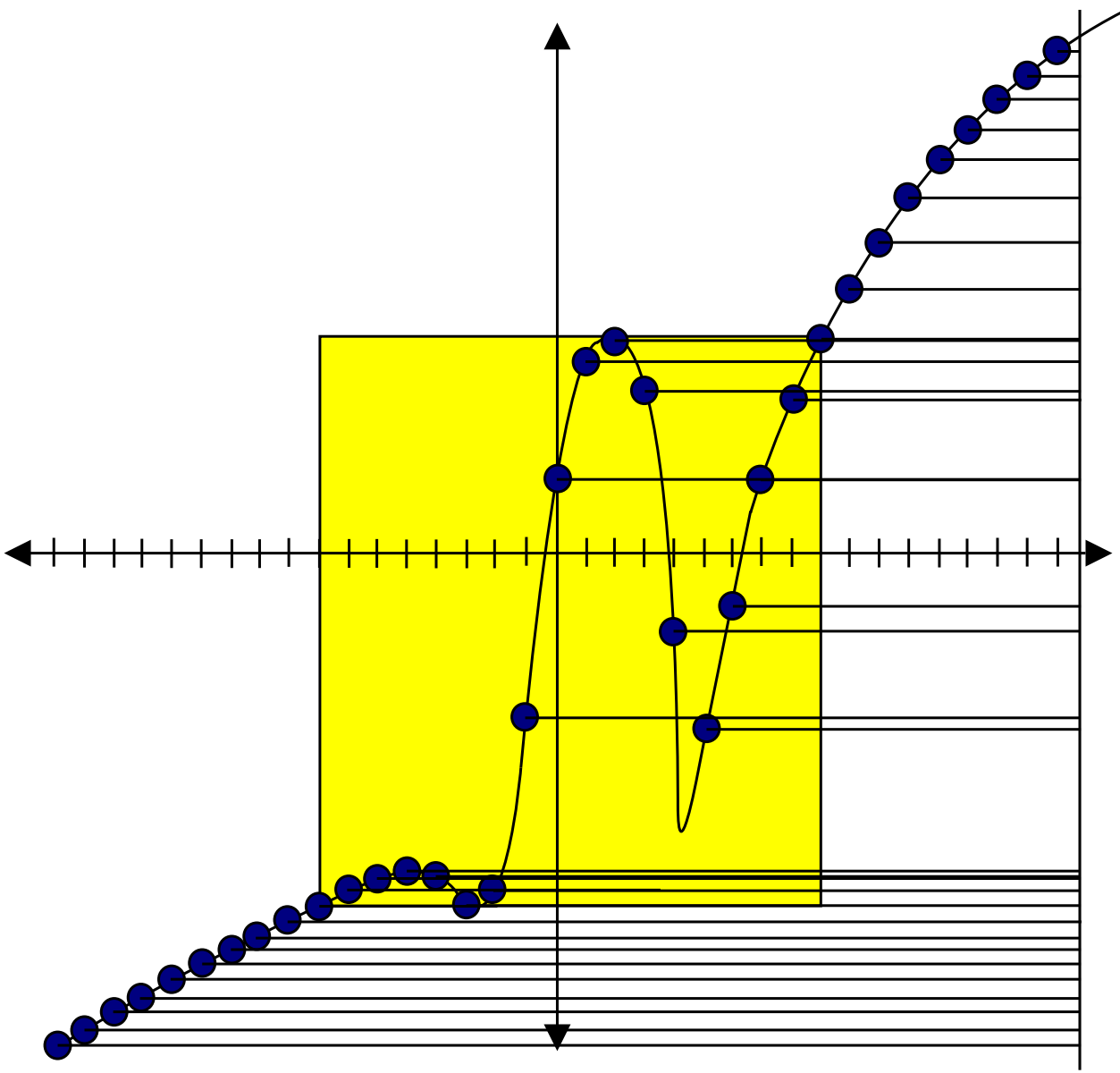
We say $f \sim g$ if for some $n \in \mathbb{Z}$, $f(x) = g(x - n)$ or $f(x) = g(-x - n)$.

Note 1: $\sim$ is an equivalence relation.

Note 2: If $f \sim g$ then $f(\mathbb{Z}) = g(\mathbb{Z})$. Converse?

Note 3: If $f \in \mathrm{Int}(\mathbb{Z})$ and $f \sim g$, then $g \in \mathrm{Int}(\mathbb{Z})$.

A Picture



Odd Degree Theorem

Thm: If f, g have the same, odd, degree and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f : [0, \infty) \to f(\mathbb{Z})$ consecutively, and $g : [0, \infty) \to f(\mathbb{Z})$ consecutively. Choose $m, n \in \mathbb{N}$ with $f(m) = g(n)$. Then $f(m + x) = g(n + x)$ for all $x \in \mathbb{N}$, so $f(m + x) = g(n + x)$.

Odd Degree Theorem

Thm: If f, g have the same, odd, degree and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f : [0, \infty) \to f(\mathbb{Z})$ consecutively, and $g : [0, \infty) \to f(\mathbb{Z})$ consecutively.
Choose $m, n \in \mathbb{N}$ with $f(m) = g(n)$. Then
$f(m + x) = g(n + x)$ for all $x \in \mathbb{N}$, so $f(m + x) = g(n + x)$.

Odd Degree Theorem

Thm: If f, g have the same, odd, degree and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f : [0, \infty) \to f(\mathbb{Z})$ consecutively, and $g : [0, \infty) \to f(\mathbb{Z})$ consecutively. Choose $m, n \in \mathbb{N}$ with $f(m) = g(n)$. Then $f(m + x) = g(n + x)$ for all $x \in \mathbb{N}$, so $f(m + x) = g(n + x)$.

Odd Degree Theorem

Thm: If f, g have the same, odd, degree and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f : [0, \infty) \to f(\mathbb{Z})$ consecutively, and $g : [0, \infty) \to f(\mathbb{Z})$ consecutively. Choose $m, n \in \mathbb{N}$ with $f(m) = g(n)$. Then $f(m + x) = g(n + x)$ for all $x \in \mathbb{N}$, so $f(m + x) = g(n + x)$.

A Picture

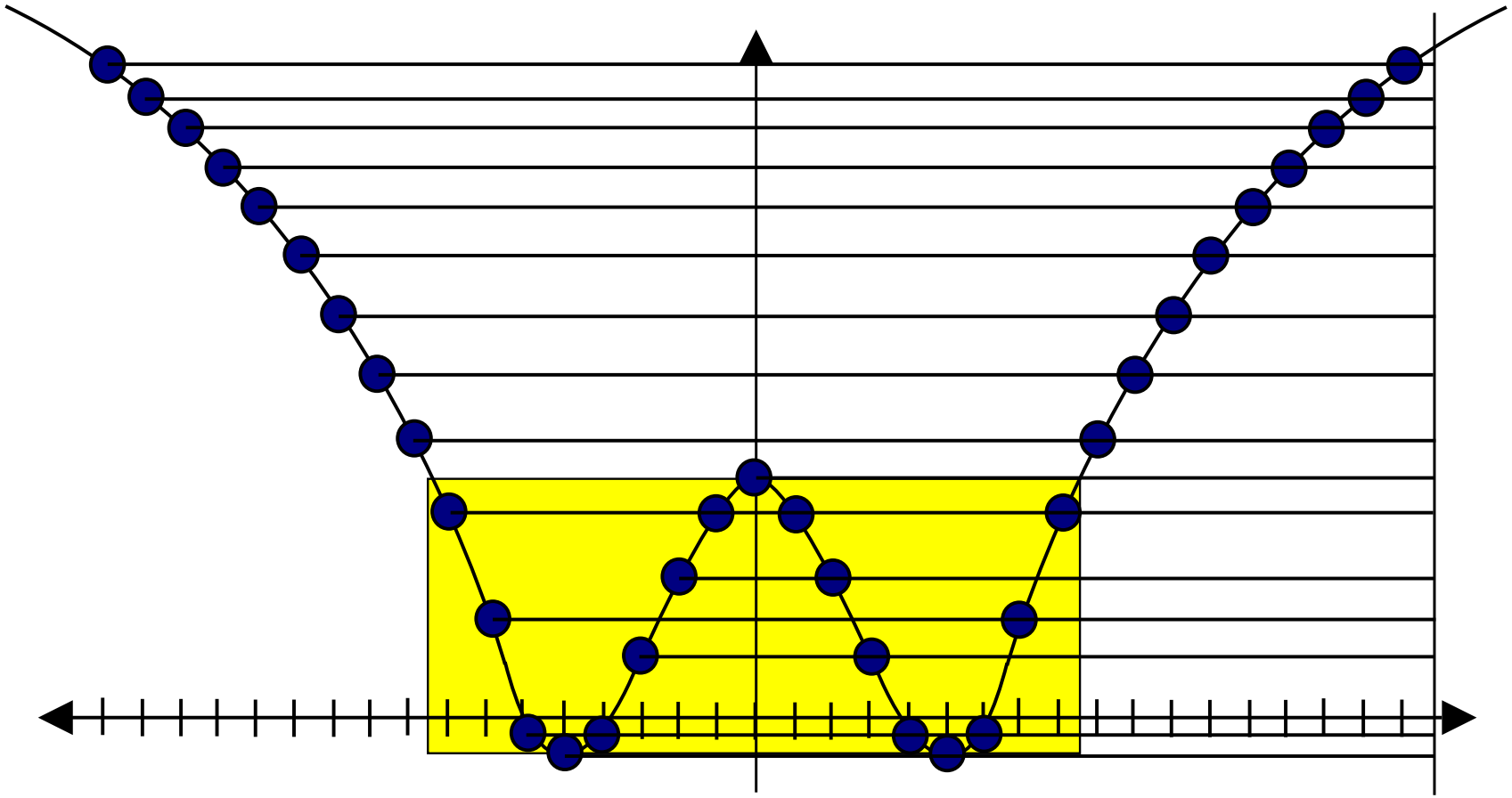


Image of left branch coincides with image of right branch

Type 1

Another Picture

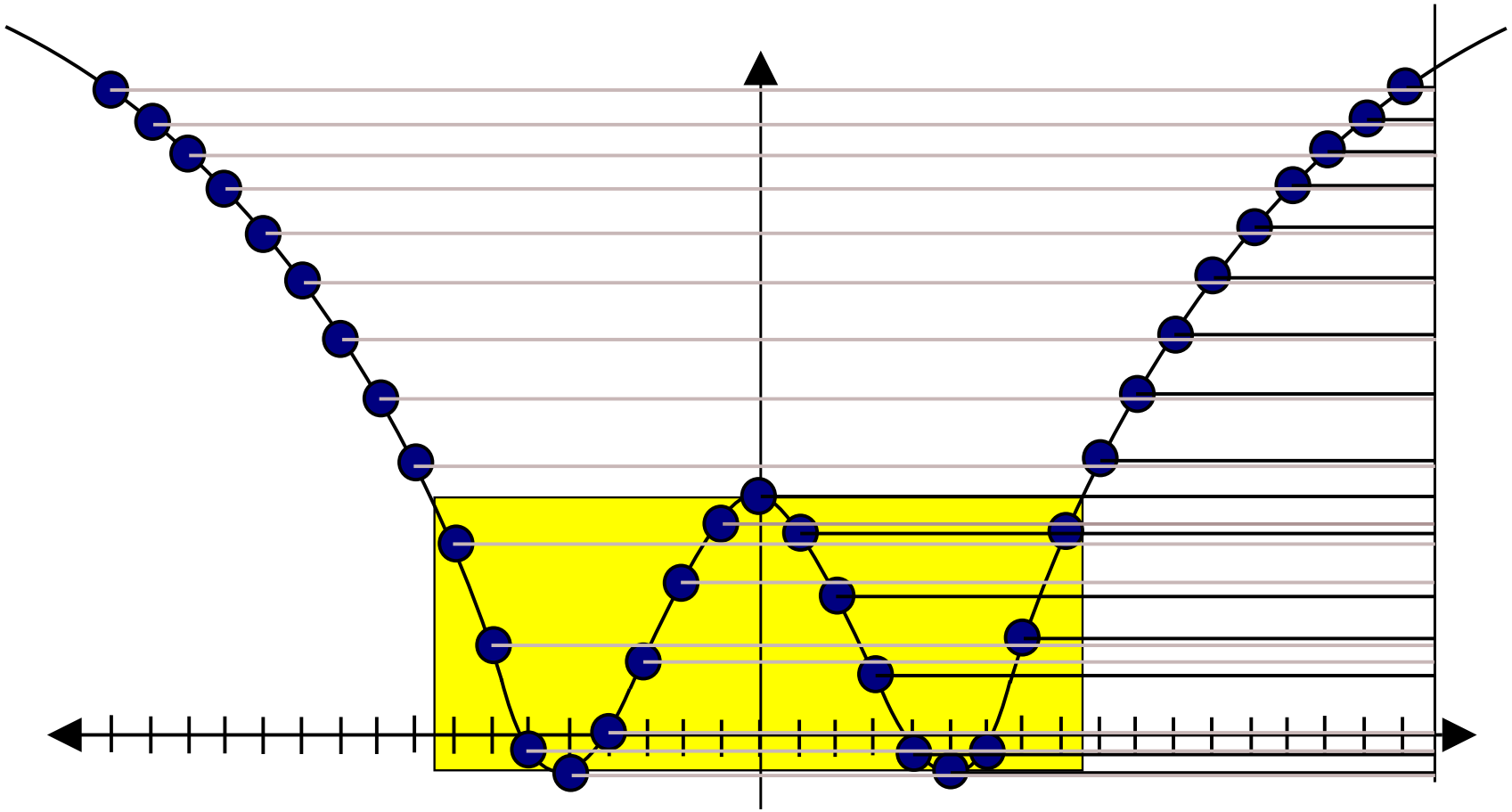


Image of left branch alternates with image of right branch

Type 2 Thm: These are the only two types.

Another Picture

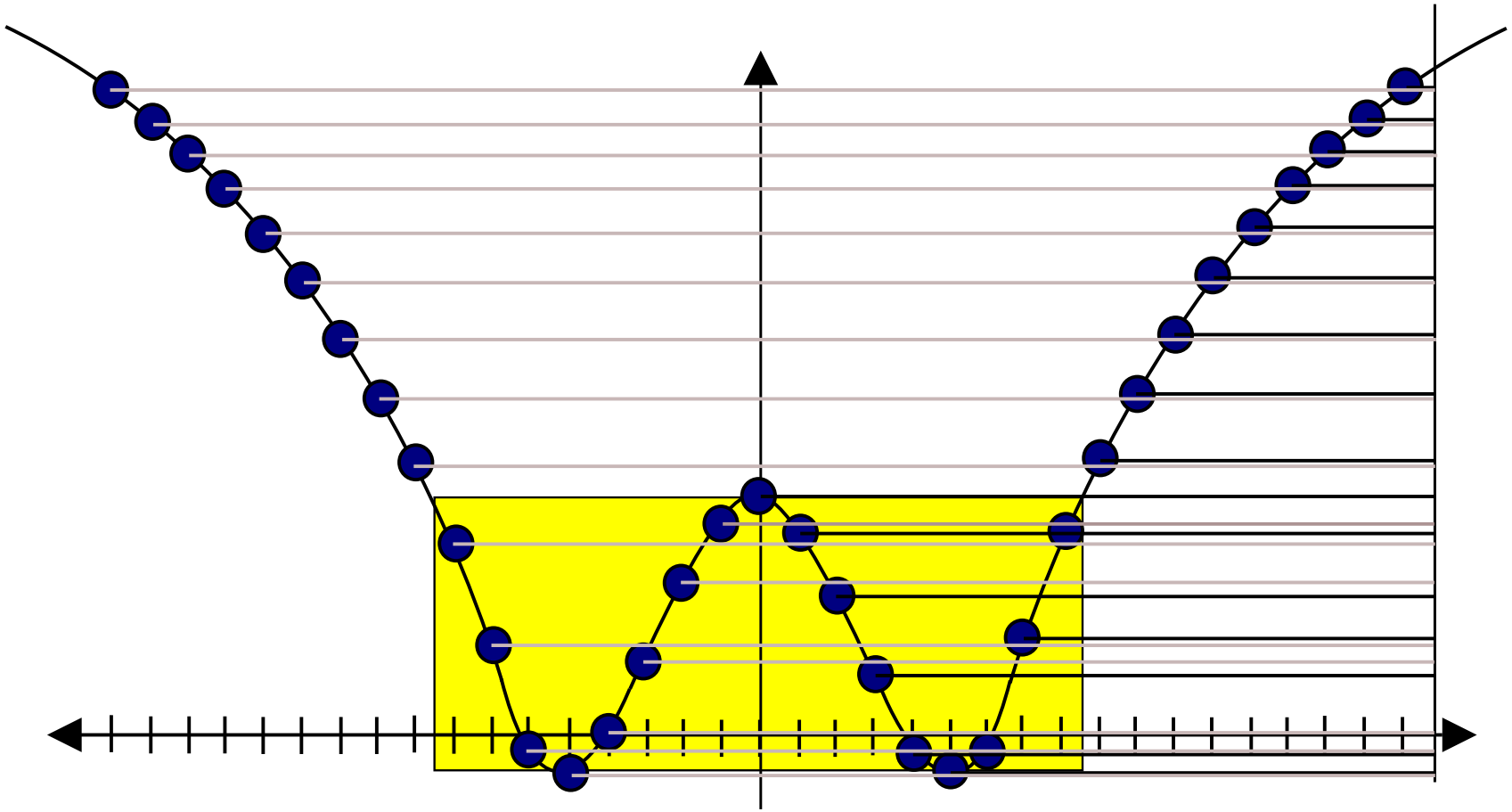


Image of left branch alternates with image of right branch

Type 2 Thm: These are the only two types.

Even Degree Theorems

Thm: If f, g have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Same as for odd degree.

Thm: If f, g have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f \sim f_{left} \sim f_{right}$ with $f_{left} : [0, \infty) \to f(\mathbb{Z})$ and $f_{right} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly g_{left}, g_{right}. Choose $m, n \in \mathbb{N}$ with $f_{left}(m) = g_{left}(n)$ [or $f_{left}(m) = g_{right}(n)$]. Then $f_{left}(m + 2x) = g_{left}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{left}(m + x) = g_{left}(n + x)$.

Even Degree Theorems

Thm: If f, g have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Same as for odd degree.

Thm: If f, g have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f \sim f_{left} \sim f_{right}$ with $f_{left} : [0, \infty) \to f(\mathbb{Z})$ and $f_{right} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly g_{left}, g_{right}. Choose $m, n \in \mathbb{N}$ with $f_{left}(m) = g_{left}(n)$ [or $f_{left}(m) = g_{right}(n)$]. Then $f_{left}(m + 2x) = g_{left}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{left}(m + x) = g_{left}(n + x)$.

Even Degree Theorems

Thm: If f, g have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Same as for odd degree.

Thm: If f, g have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f \sim f_{left} \sim f_{right}$ with $f_{left} : [0, \infty) \to f(\mathbb{Z})$ and $f_{right} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly g_{left}, g_{right}. Choose $m, n \in \mathbb{N}$ with $f_{left}(m) = g_{left}(n)$ [or $f_{left}(m) = g_{right}(n)$]. Then $f_{left}(m + 2x) = g_{left}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{left}(m + x) = g_{left}(n + x)$.

Even Degree Theorems

Thm: If f, g have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Same as for odd degree.

Thm: If f, g have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f \sim f_{left} \sim f_{right}$ with $f_{left} : [0, \infty) \to f(\mathbb{Z})$ and $f_{right} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly g_{left}, g_{right}. Choose $m, n \in \mathbb{N}$ with $f_{left}(m) = g_{left}(n)$ [or $f_{left}(m) = g_{right}(n)$]. Then $f_{left}(m + 2x) = g_{left}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{left}(m + x) = g_{left}(n + x)$.

Even Degree Theorems

Thm: If f, g have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Same as for odd degree.

Thm: If f, g have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

Proof: Without loss of generality, $f \sim f_{left} \sim f_{right}$ with $f_{left} : [0, \infty) \to f(\mathbb{Z})$ and $f_{right} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly g_{left}, g_{right}. Choose $m, n \in \mathbb{N}$ with $f_{left}(m) = g_{left}(n)$ [or $f_{left}(m) = g_{right}(n)$]. Then $f_{left}(m + 2x) = g_{left}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{left}(m + x) = g_{left}(n + x)$.

Four Types of Even-Degree Polynomials

A polynomial has at most one line of reflection $x = k$:

(•) $k \in \mathbb{Z}$

(•) $2k \in \mathbb{Z}$, $k \notin \mathbb{Z}$

(•) $4k \in \mathbb{Z}$, $2k \notin \mathbb{Z}$

(•) other, including no line of reflection

Four Types of Even-Degree Polynomials

A polynomial has at most one line of reflection $x = k$:

(•) $k \in \mathbb{Z}$

(•) $2k \in \mathbb{Z}$, $k \notin \mathbb{Z}$

(•) $4k \in \mathbb{Z}$, $2k \notin \mathbb{Z}$

(•) other, including no line of reflection

Four Types of Even-Degree Polynomials

A polynomial has at most one line of reflection $x = k$:

(1a) $k \in \mathbb{Z}$

(1b) $2k \in \mathbb{Z}$, $k \notin \mathbb{Z}$

(2a) $4k \in \mathbb{Z}$, $2k \notin \mathbb{Z}$

(2b) other, including no line of reflection

Thm: (1a),(1b) are of Type 1. (2a),(2b) are of Type 2.

Cross-Type Image Sharing

Thm: Suppose f, g have the same, even, degree with $f(\mathbb{Z}) = g(\mathbb{Z})$. Suppose f is of Type 1, g is of Type 2. Then f is of Type (1b), g is of Type (2a), and $f(2x) \sim g(x)$

Thm: $\phi : \text{Type } (1b) \to \text{Type } (2a)$ via $\phi(f(x)) = f(2x)$ is a bijection.

Cross-Type Image Sharing

Thm: Suppose f, g have the same, even, degree with $f(\mathbb{Z}) = g(\mathbb{Z})$. Suppose f is of Type 1, g is of Type 2. Then f is of Type (1b), g is of Type (2a), and

$f(2x) \sim g(x)$

Thm: $\phi : \text{Type } (1b) \to \text{Type } (2a)$ via $\phi(f(x)) = f(2x)$ is a bijection.

Cross-Type Image Sharing

Thm: Suppose f, g have the same, even, degree with $f(\mathbb{Z}) = g(\mathbb{Z})$. Suppose f is of Type 1, g is of Type 2. Then f is of Type (1b), g is of Type (2a), and $f(2x) \sim g(x)$

Thm: $\phi : \text{Type } (1b) \to \text{Type } (2a)$ via $\phi(f(x)) = f(2x)$ is a bijection.

Cross-Type Image Sharing

Thm: Suppose f, g have the same, even, degree with $f(\mathbb{Z}) = g(\mathbb{Z})$. Suppose f is of Type 1, g is of Type 2. Then f is of Type (1b), g is of Type (2a), and $f(2x) \sim g(x)$

Thm: $\phi : \text{Type } (1b) \to \text{Type } (2a)$ via $\phi(f(x)) = f(2x)$ is a bijection.