Sums of Generalized Tetrahedral Numbers

A longstanding, famous, problem in number theory is to determine how many perfect powers one needs to sum to a given integer. If we insist that the perfect powers share the same exponent, and are all positive, this is called Waring's problem. For example, it has been known for over 100 years (see [**2**]) that it takes nine positive cubes to represent every positive integer. If instead we allow the cubes to be positive and negative, this is a different question, with many cases still open. For example, only in 2020 it was found (in [**1**]) that 42 is expressible as the sum of three cubes of integers.

We turn now from perfect powers to another type of figurate numbers. The nth tetrahedral number $Te_n = \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$, a cubic polynomial in n, represents the sum of the first n triangular numbers. A 19th century conjecture of Pollock is that every positive integer may be expressed as the sum of at most five (positive) tetrahedral numbers.

Recently attention has been given (in [**3**]) to a generalized version of Pollock's conjecture, where n may be any integer. There it was proved that all integers are expressible as the sum of at most four generalized tetrahedral numbers. Below we show that two generalized tetrahedral numbers do not suffice. Can every integer be expressed as the sum of three generalized tetrahedral numbers?

Lemma. *If $p \notin \{2, 5, 11\}$ is prime, it is not the sum of two tetrahedral numbers.*

Proof. Suppose that $p = Te_n + Te_{n-k}$ for some integers $n \geq k \geq 0$. This may be rearranged as $6p = (2n-k+2)(k^2 - kn - k + n^2 + 2n)$. If $2n - k + 2$ divides 6, then $(n, k) \in \{(0,0), (1,1), (2,0), (3,2), (4,4)\}$. Otherwise, we must have $k^2 - kn - k + n^2 + 2n = \alpha$ for some $\alpha | 6$. If $\alpha < -1$ there are no solutions; if $\alpha \geq -1$ this is a skew ellipse entirely contained in the halfplane $n \leq 3$. Hence there is just a small set of (n, k) pairs to check. ■

Theorem. *If p is an odd prime and $(p \mod 5) = 2$, then p is not the sum of two generalized tetrahedral numbers.*

Proof. By the Lemma, p is not the sum of two positive tetrahedral numbers. Suppose that $p = Te_n + Te_{-(n-k)}$ for some integers $n \geq k \geq 0$ with $p = Te_n + Te_{-(n-k)}$. This may be rearranged as $6p = (k+2)(3n^2 - 3kn + (k^2 + k))$. If $(k+2)|6$, then $k \in \{0, 1, 4\}$. If $k = 0$ then $p = n^2$. If $k = 1$ then we calculate $(3n^2 - 3n + 2 \mod 5) \in \{0, 2, 3\}$ but $(2p \mod 5) = 4$. If $k = 4$ then we calculate $(3n^2 - 12n + 20 \mod 5) \in \{0, 1, 3\}$ but $(p \mod 5) = 2$. Last is the case $3n^2 - 3kn + k^2 + k = \alpha$ for some positive $\alpha | 6$. This is a skew ellipse in the halfplane $n \leq 3$, and again this gives only a few (n, k) pairs to check. ■

REFERENCES

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—Submitted by Vadim Ponomarenko, San Diego State University

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