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Surprises in Knockout Tournaments

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The NCAA Division I basketball tournament has 64 teams participate, in $6 = \log_2 64$ rounds, with no ties permitted and the loser of each two-player match eliminated. This is one well-known example of a knockout or single-elimination tournament. Other examples are most of the Wimbledon events (indeed all large tennis tournaments), and the NFL playoffs. Knockout tournaments are of substantial interest to mathematicians and other scientists (see, e.g., [**1, 5, 7**]). These are distinct from round-robin tournaments, which are of even greater interest (see, e.g., [**2, 3, 6**]), but which we will not consider here.

The (knockout) tournament winner ends unbeaten, and the tournament runner-up loses only in the final match. The tournament rankings of the other participants are unclear, so we will focus only on the top two possible tournament outcomes. The overall top player, named Frankie, is more likely to finish first than second, being favored in the final match. Our interest lies in the second-best player, named Skylar.

Intuition might suggest that Skylar should be more likely to finish second than first. However, this is not always the case. For example, in the aforementioned basketball tournament over the fifteen years 2004-2018, the men's second seed ended up winning three times, and finishing second twice. Of course, this is a small sample size. We call a situation where Skylar is more likely to finish first than second a *surprise*, and we will characterize such surprises, under certain assumptions.

Our main assumption is that all other tournament participants, named Player, are indistinguishable from each other. Hence, there are just three probabilities of interest. Set p to be the probability that Skylar beats Frankie. Set q to be the probability that Skylar beats Player. Set r to be the probability that Frankie beats Player. Our probabilities should satisfy $0 \leq p < \frac{1}{2} < q \leq r \leq 1$. For convenience, set $p' = 1-p, q' = 1-q, r' = 1-r$. Our tournament will have n rounds before the championship, $n+1$ total rounds. We assume that there are no byes, i.e. there are 2^{n+1} players.

The two most common ways of designing a knockout tournament are by seeding the players, and randomly. The difference between the two might not be very large (see [**4**]), but we will treat these two cases separately.

Seeded Knockout Tournaments

If the tournament is seeded, then Frankie and Skylar can only meet in the final round. The probability that Skylar finishes first is $q^n r^n p + q^n(1-r^n)q$, where the first term corresponds to facing Frankie in the finals, and the second corresponds to Frankie losing before the finals. The probability that Skylar finishes second is $q^n r^n p' + q^n(1-r^n)q'$.

The surprise happens when the difference $q^n(r^n p + (1-r^n)q - r^n p' - (1-r^n)q') > 0$. We cancel the positive q^n, and note that $p - p' = 2p-1 < 0$ while $q - q' = 2q - 1 > 0$. We then combine terms to get $r^n(2p-1) + (1-r^n)(2q-1) > 0$, and rearrange to $(2q-1) - 2r^n(q-p) > 0$.

Hence, the surprise happens in a seeded tournament exactly when Condition (S) holds:

$$2r^n < \frac{2q-1}{q-p} = 2 - \frac{1-2p}{q-p} \tag{S}$$

Condition (S) has some interesting properties. If it holds for a particular p, q, r, n, then it will still hold if we increase any one of n, p, or q (while holding all other variables fixed), or if we decrease r. For any fixed p, q, r, with $r < 1$, there is some minimum n which will ensure that the surprise occurs for that and all larger n. Setting $p = 0$ we get $2r^n < 2 - \frac{1}{q}$; if this condition holds, then the surprise will happen for all p. Setting $r = 1$, we get $2 < 2 - \frac{1-2p}{q-p}$, which never holds. Also, consider the limit as $q, r \to \frac{1}{2}$ (with fixed p, n), the surprise again never holds.

For the special case $q = r$, we can plot the surprise curves for various p, q, n. The area above each curve is the surprise region, where Condition (S) holds. Each curve passes through $(0.5, 0.5)$ and $(1, 0.5)$ because no surprise can happen for those values of r. Note that increasing q might exit the surprise region; this is because we are simultaneously increasing r.

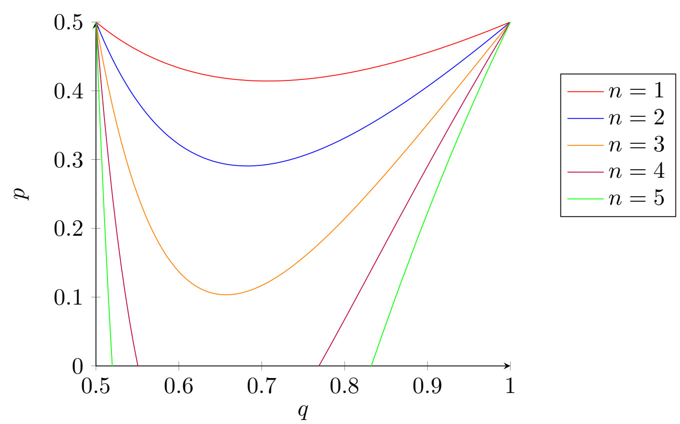

Figure 1 Surprise curves for the special case $q = r$, seeded tournament.

Random Knockout Tournaments

For random knockout tournaments, the computation is more difficult. Frankie and Skylar might meet in the finals, or in round k, for $1 \le k \le n$. If they meet in round k, then Frankie must have won up to that point, in a field of 2^{k-1} players. If they meet in the finals, then Frankie must have already won in his half-tournament of 2^n players. However, these $n+1$ possibilities of when the players could meet are not equally likely. Set $m = 2^{n+1} - 1$. In a random tournament, the m positions other than Skylar's are all equally likely for Frankie. In 2^{k-1} of them, Frankie could meet Skylar in round k, and in 2^n of them, they could meet in the finals.

Hence, the probability that Skylar finishes first is

$$\sum_{k=1}^{n+1} \frac{2^{k-1}}{m} q^n r^{k-1} p + \sum_{k=1}^{n+1} \frac{2^{k-1}}{m} q^n (1 - r^{k-1}) q. \tag{Skylar\#1}$$

Note that the first sum corresponds to Skylar beating Frankie at some point (including in the finals), while the second sum corresponds to Frankie losing before facing Skylar.

The probability that Skylar finishes second is

$$\frac{2^n}{m}q^n r^n p' + \sum_{k=1}^{n} \frac{2^{k-1}}{m} q^{n-1} r^{k-1} p q' + \sum_{k=1}^{n+1} \frac{2^{k-1}}{m} q^n (1 - r^{k-1}) q' \qquad \text{(Skylar\#2)}$$

Here, the first term corresponds to Skylar losing to Frankie in the finals. The second term corresponds to Skylar beating Frankie in round k, but losing in the finals. The third term corresponds to Frankie losing prior to meeting Skylar, and Skylar winning every round except the finals. The surprise happens when (Skylar#1) is greater than (Skylar#2). We multiply each by the positive $\frac{m}{q^{n-1}}$, and pull constants out of sums, to find the surprise equivalent to

$$qp \sum_{k=1}^{n+1} 2^{k-1} r^{k-1} + q^2 \sum_{k=1}^{n+1} 2^{k-1}(1 - r^{k-1}) >$$
$$2^n q r^n p' + pq' \sum_{k=1}^{n} 2^{k-1} r^{k-1} + qq' \sum_{k=1}^{n+1} 2^{k-1}(1 - r^{k-1})$$

Each series is either geometric or the difference of two geometric series, hence we can find their sums. After considerable rearrangement, we find the surprise happens in a random tournament exactly when Condition (R) holds:

$$2^{n+1} q(2q-1)(2r-1) + (2q-1)(2q-2qr-p) > 2^n r^n (q-p)(4qr-1) \quad \text{(R)}$$

Note that $2q-1, 2r-1, q-p, 4qr-1$ are each positive. Considering the bounds on p, q, r, we find that $-\frac{1}{2} \leq 2q - 2qr - p \leq \frac{1}{2}$. Condition (R) shares some properties with Condition (S). For convenience, set $f(p,q,r,n) = 2^{n+1}q(2q-1)(2r-1) + (2q-1)(2q-2qr-p) - 2^n r^n (q-p)(4qr-1)$; Condition (R) is equivalent to $f(p,q,r,n) > 0$.

We calculate $f(p+\epsilon, q, r, n) - f(p,q,r,n) = -(2q-1)\epsilon + 2^n r^n (4qr-1)\epsilon \geq \epsilon(2q-1)(-1 + 2^n r^n (2r + \frac{2r-1}{2q-1})) > 0$. Hence, if Condition (R) holds, it will still hold if we increase p.

We calculate $f(p,q,r,n+1) - 2rf(p,q,r,n) = 2^{n+1}q(2q-1)(2r-1)(2-2r) + (1-2r)(2q-1)(2q-2qr-p) = (2r-1)(2q-1)(q(1-r)(2^{n+2}-2) + p) \geq 0$. Hence, if Condition (R) holds, it will still hold if we increase n. Further, if $r < 1$ then $\lim_{n\to\infty} \frac{f(p,q,r,n)}{2^n} = 2q(2q-1)(2r-1) > 0$, so there is some minimum n which will ensure that the surprise occurs for that and all larger n.

Hence, if the surprise holds for $p = 0, n = 1$, then it will hold independently of p, n. Unfortunately, no values of q, r meet this condition; neither for $p = 0, n = 2$. However, for $p = 0$ and for each $n \geq 3$, there is a region in the $q - r$ plane for which the surprise will hold independently of p. Note that each successive region includes all previous ones. These regions are plotted in Figure 2.

Unfortunately, varying q or r does not appear to respect Condition (R) in the same way as with Condition (S). We are able to prove that, considering each of these variables separately, the surprise will hold on a (possibly empty) interval.

The function $f(p,q,r,n)$, fixing p, r, n, is a quadratic polynomial in q, with leading coefficient $2^{n+2}(2r-1) + 4(1-r) - 2^{n+2}r^{n+1}$. This leading coefficient is positive

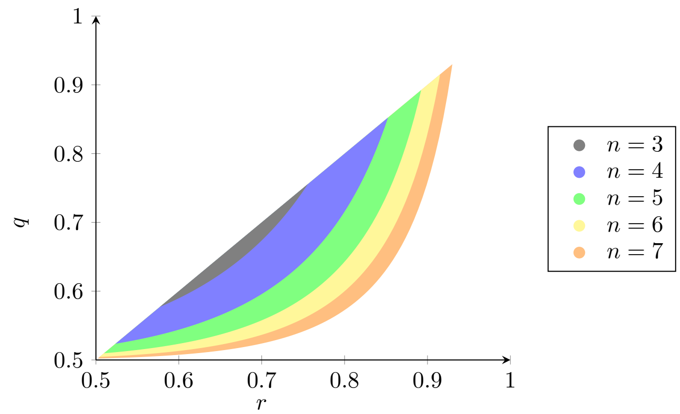

Figure 2 Surprise regions for $p = 0$, random tournament.

for all $r \in (0.5, 1)$, so the parabola points up. Hence, $f(p, q, r, n) > 0$ will hold for all $q \in \mathbb{R}$, apart from some interval. This interval may interesect with (or contain all of) $(0.5, r]$.

Considering $f(p, q, r, n)$ as a function of r, we find that $\frac{\partial}{\partial r} f(p, q, r, n) = Ar^n + B$, for some real constants A, B. This has at most one positive zero. By the mean value theorem, $f(p, q, r, n)$ has at most one positive zero. Hence the surprise will happen for r in some halfline intersected with $[q, 1]$.

As before, we consider the special case of $q = r$. We have $\lim_{r\to 1/2} \frac{f(p,q,r,n)}{2r-1} = \lim_{r\to 1/2} 2^{n+1}r(2r - 1) + 2r(1 - r) - p - 2^n r^n(r - p)(2r + 1) = 1/2 - p - (1/2 - p)(2) = -1/2 + p < 0$. Hence, for q, r sufficiently close to $1/2$, Condition (R) fails to hold. On the other hand, if we take $q = r = 1$, then Condition (R) simplifies to $p > \frac{2^n}{3\cdot 2^n - 1}$. Compare with the seeded tournament case, where for $r = 1$ the surprise is impossible. We plot the surprise curves below.

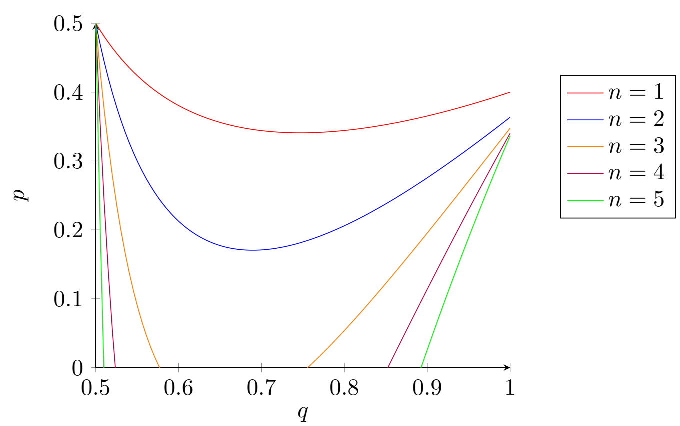

Figure 3 Surprise curves for the special case $q = r$, random tournament.

We close by asking if a player even lower-ranked than second, might still be more likely to win the tournament than to finish second. Such a situation would be an even greater surprise.

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