Factorizations Common to Some Subsemigroups of $(\mathbb{N},+)$ and $(\mathbb{N},\times)$

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Iberian Meeting on Numerical Semigroups
Granada February 4, 2010

http://www-rohan.sdsu.edu/~vadim/granada.pdf

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Main Result, very broadly

Theorem: Given M, a subsemigroup of $(\mathbb{N}, \times)$, and $m \in \mathbb{N}$.

Then there is a "partial homomorphism" $f : M \to (\mathbb{N}, +)$.

f is a homomorphism between $M \cap [1, m]$ and its image.
i.e., $f(xy) = f(x) + f(y)$ provided $\{xy, x, y\} \subseteq M \cap [1, m]$
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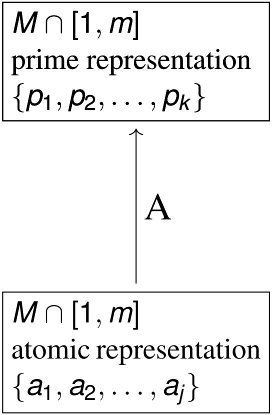
Main Result, in a picture

$M \cap [1, m]$
atomic representation
$\{a_1, a_2, \ldots, a_j\}$

Main Result, in a picture



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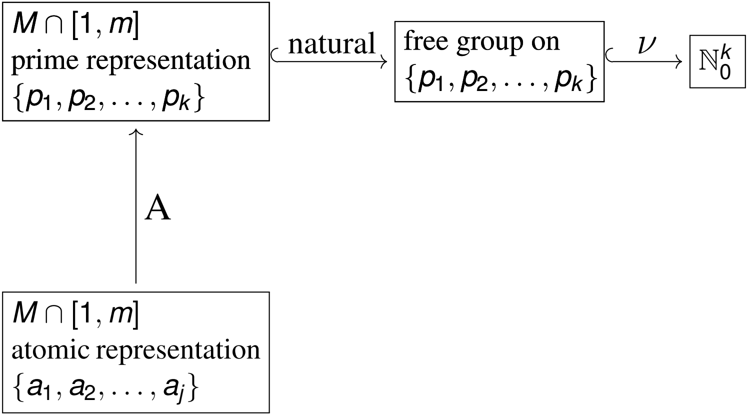

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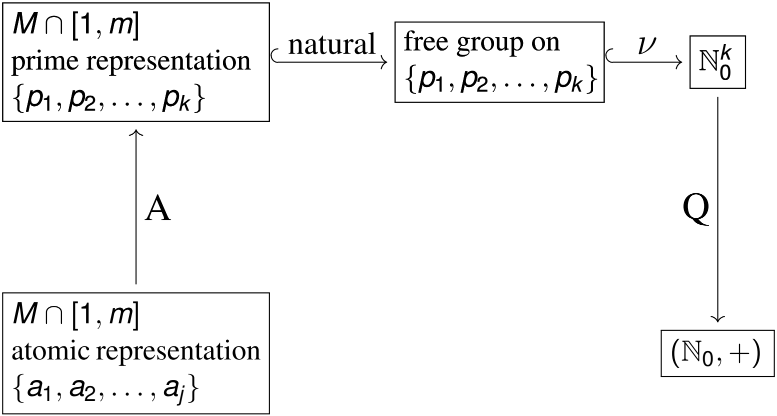

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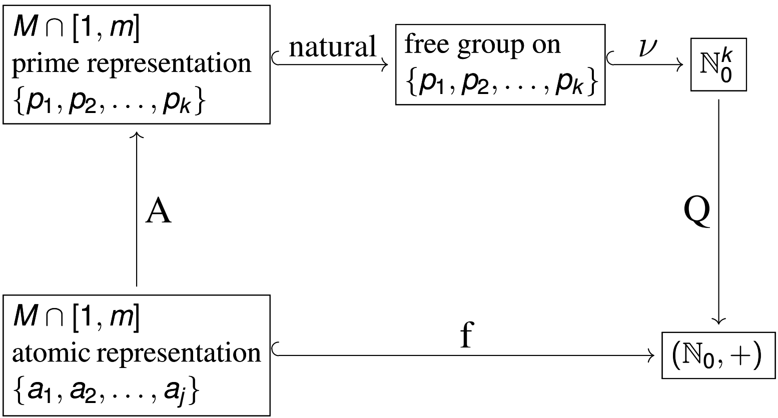

Trivial but Important Lemma

Lemma: Consider $x_0, x_1, \ldots, x_{k-1} \in \mathbb{Z}$ and $q \in \mathbb{N}$.
Suppose that $x_0 q^0 + x_1 q^1 + x_2 q^2 + \cdots + x_{k-1} q^{k-1} = 0$.
Then $x_0 = x_1 = \cdots = x_{k-1} = 0$, or $|x_i| \geq q$ for some i.

Proof: Assume k is minimal to give a counterexample. All terms but the last: $|x_0 q^0 + x_1 q^1 + \cdots + x_{k-2} q^{k-2}| \leq (q-1)(q^0 + q^1 + \cdots + q^{k-2}) = (q-1)\frac{q^{k-1}-1}{q-1} < q^{k-1}$.
Hence $x_{k-1} = 0$, and k was not minimal.

Consequence: Choose sufficiently large q, then take $Q = [1, q, q^2, \ldots, q^{k-1}]$.

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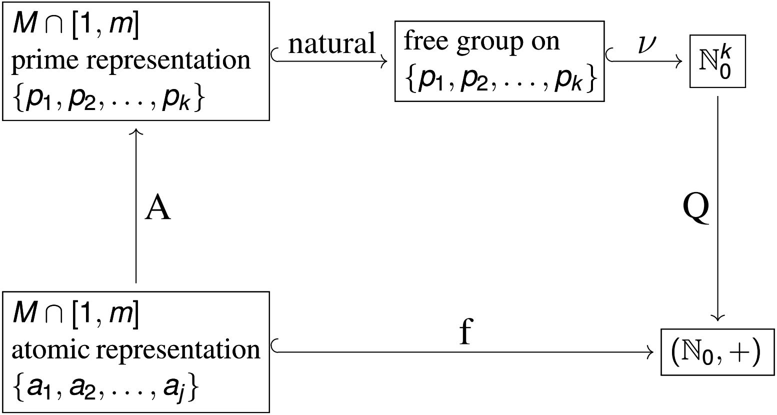
Main Result, one more time

$Q = [1, q, q^2, \ldots, q^{k-1}]$; $Qx = 0$ implies $x = 0$ or $||x||_\infty \geq q$.



A Detailed Example

Let $M =< 144, 216, 108, 162 > \subseteq (\mathbb{N}, \times)$

$j = 4$; $144 = 2^4 3^2, 216 = 2^3 3^3, 108 = 2^2 3^3, 162 = 2^1 3^4$

$k = 2$; $p_1 = 2, p_2 = 3$

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix}$$

Q1: Find all factorizations of $(216)^{10}$.

Q2: Find all factorizations of $(216)^{20}$.

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$M =< 144, 216, 108, 162 >, \; m = (216)^{10}$

$$(144)^{10} \le (216)^{10}, \quad (144)^{11} > (216)^{10}, \quad (144)^{10} = 2^{40}3^{20}$$
$$(216)^{10} \le (216)^{10}, \quad (216)^{11} > (216)^{10}, \quad (216)^{10} = 2^{30}3^{30}$$
$$(108)^{11} \le (216)^{10}, \quad (108)^{12} > (216)^{10}, \quad (108)^{11} = 2^{22}3^{33}$$
$$(162)^{10} \le (216)^{10}, \quad (162)^{11} > (216)^{10}, \quad (162)^{10} = 2^{10}3^{40}$$

We take $q = \max\{40, 20, 30, 30, 22, 33, 10, 40\} + 1 = 41.$

Simpler strategy:
$2^{77} \le (216)^{10} \quad 2^{78} > (216)^{10}, 3^{48} \le (216)^{10} \quad 3^{49} > (216)^{10}$
We could take the larger $q = \max\{77, 48\} + 1 = 78.$

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$q = 41, k = 2$, so $Q = \begin{pmatrix} 1 & 41 \end{pmatrix}$.

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix}$$

$$QA = \begin{pmatrix} 86 & 126 & 125 & 165 \end{pmatrix}$$

Hence we consider $N =< 86, 126, 125, 165 > \subseteq (\mathbb{N}, +)$.

We factor $1260 = [0, 10, 0, 0]$ in N. GAP+numericalsgps factors: $[0, 10, 0, 0], [3, 2, 6, 0], [4, 1, 5, 1], [5, 0, 4, 2]$

$(216)^{10}$, $(144)^3(216)^2(108)^6$, $(144)^4(216)(108)^5(162)$, $(144)^5(108)^4(162)^2$

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We turn now to $m = (216)^{20}$.

NOTE: Our previous semigroup $< 86, 126, 125, 165 >$ based on $q = 41$ does not work.

GAP factors $[0, 20, 0, 0]$ as (among others) $[0, 0, 3, 13]$ but $(216)^{20} \neq (108)^3(162)^{13}$

Set $p(n) = n^{\left\lfloor \frac{\log(m)}{\log(n)} \right\rfloor}$. $p(144) = 2^{84}3^{42}$, $p(216) = 2^{60}3^{60}$, $p(108) = 2^{44}3^{66}$, $p(162) = 2^{21}3^{84}$

We take $q = \max\{84, 42, 60, 60, 44, 66, 21, 84\} + 1 = 85$.

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$QA = \begin{pmatrix} 174 & 258 & 257 & 341 \end{pmatrix}$

Hence we consider $N =< 174, 258, 257, 341 > \subseteq (\mathbb{N}, +)$.

We factor $5160 = [0, 20, 0, 0]$ in N.

GAP+numericalsgps instantly gives the 13 factorizations:

$[0, 20, 0, 0], [3, 12, 6, 0], [4, 11, 5, 1], [5, 10, 4, 2],$
$[6, 9, 3, 3], [6, 4, 12, 0], [7, 8, 2, 4], [7, 3, 11, 1],$
$[8, 7, 1, 5], [8, 2, 10, 2], [9, 6, 0, 6], [9, 1, 9, 3], [10, 0, 8, 4]$

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Note: $q = 85$ equally works for all of $[1, (216)^{20}]$.

For example, back to Q1, to factor $(216)^{10} = [0, 10, 0, 0]$.
GAP factors $2580 = [0, 10, 0, 0]$ in the same four ways.

2580: catenary degree=9, elasticity=$\frac{11}{10}$, delta set=$\{1\}$.

elasticity of N is $\frac{341}{174} \geq$ elasticity of $M \cap [1, m]$.
catenary degree of N is $29 \geq$ catenary deg. of $M \cap [1, m]$.

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Genesis

Chapman, Herr, Rooney

"A Factorization Formula for Class Number Two", 1999

$M \cong < 4, 9, 25, 6, 10, 15 >$, $m = 900 = 4 \cdot 15^2$.

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix}$$

Calculate $q = 9$ as before, take $Q = [13, 118, 1063]$ (differs from previous Q for technical reasons)

$N =< 26, 236, 2126, 131, 1076, 1181 >$

$2388 = [1, 0, 0, 0, 0, 2]$ has five factorizations.

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