Summation Using an Expected Value Identity

Let X be a random variable taking values among the nonnegative integers. We recall the well-known identity $\mathbb{E}[X] = \sum_{n\geq 0} \mathbb{P}(X > n)$. For convenience, we set $p_t = \mathbb{P}(X = t)$, arriving at the beautiful identity

$$\sum_{x=0}^{\infty} x p_x = \sum_{x=0}^{\infty} \left(1 - \sum_{y=0}^{x} p_y\right). \tag{1}$$

If we further insist that X has support in $[0, n]$, we may rearrange this to get

$$n + 1 = \sum_{x=0}^{n} 1 = \sum_{x=0}^{n} \left(x p_x + \sum_{y=0}^{x} p_y\right). \tag{2}$$

We now offer two applications of (2). First, take the uniform distribution for X, namely $p_x = \frac{1}{n+1}$. This gives

$$n + 1 = \sum_{x=0}^{n} \left(\frac{x}{n+1} + \sum_{y=0}^{x} \frac{1}{n+1}\right) = \sum_{x=0}^{n} \frac{x}{n+1} + \frac{x+1}{n+1} = 1 + \frac{2}{n+1} \sum_{x=0}^{n} x.$$

We may rearrange this to find a new proof of the familiar $\sum_{x=0}^{n} x = \frac{n(n+1)}{2}$. Compare this also to the probabilistic proof in [**1**]. For our second application, take the distribution $p_x = \frac{2x}{n(n+1)}$. Applying (2) gives

$$n + 1 = \sum_{x=0}^{n} \left(\frac{2x^2}{n(n+1)} + \sum_{y=0}^{x} \frac{2y}{n(n+1)}\right) = \frac{1}{n(n+1)} \sum_{x=0}^{n} 2x^2 + x(x+1).$$

We rearrange this to $3\sum_{x=0}^{n} x^2 = n(n+1)^2 - \sum_{x=0}^{n} x$, which gives a new proof of the familiar $\sum_{x=0}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$.

REFERENCES

1. Treviño, E. (2019). Probabilistic Proof that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. *Amer. Math. Monthly*. 126 (9): 840.

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