

# A New Factorization Invariant

## diversity in algebra

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<http://www-rohan.sdsu.edu/~vadim/diversity.pdf>



## Motivation

Suppose  $p|ab$ , in some [semigroup/monoid/domain].

$p$  *prime* implies  $(p|a \text{ or } p|b)$ . [UFM/UFD]

$p$  *primary* implies  $(p|a \text{ or } p|b \text{ or } (p \upharpoonright a \text{ and } p \upharpoonright b))$ ,

where  $x \upharpoonright y$  means  $x|y^n$  for some  $n \in \mathbb{N}$ . [WFM/WFD]

$p$  *almost primary* implies  $(p \upharpoonright a \text{ or } p \upharpoonright b)$ . [?]

Ex:  $2\mathbb{Z}$  6 is almost primary, not primary:  $6|2 \cdot 18$



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## Almost-primary factorizations

A factorization into almost primary elements  $p_1 p_2 \cdots p_k$  is *reduced* if  $p_i \not\sim p_j$ , for every  $p_i, p_j$ .

Note: If  $p_i \mid p_j$ , then  $(p_i p_j)$  is almost primary.

### Theorem 1

If  $x$  has any factorization into almost primary elements, then it has a reduced factorization into almost primary elements, unique up to order and weak associates ( $p \mid q$  and  $q \mid p$ ).

### Theorem 2

Suppose the [monoid/domain] is atomic. All elements have such a unique factorization, if and only if all atoms are almost primary.



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## The new invariant

Given [monoid/domain]  $X$ , with  $x \in X$ , we define the diversity of  $x$  as the largest cardinality of  $S \subseteq X$ , where  $x \mid (\prod S)$ , but  $x \nmid (\prod \text{any proper subset of } S)$ .  
We write  $\text{div}(x)$ .

### Properties:

$$\text{div}(xy) \leq \text{div}(x) + \text{div}(y)$$

$\text{div}(x) = 1$  if and only if  $x$  is almost primary (or a unit)

diversity  $\leq$  tame degree

diversity independent of catenary degree





## An example

$\mathbb{Z} \setminus \{0, -1\}$ , under multiplication. 1 is the only unit.

Atoms are  $\pm$  rational primes.

Atoms almost primary, but not primary:  $2 \mid (-2) \cdot 3$ .

Atoms are *homogeneous*: For atom  $p$ , if  $q \mid p$ , then  $p \mid q$ .



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# Shameless advertising

Please encourage your students to apply to the  
San Diego State University Mathematics REU.

<http://www.sci.sdsu.edu/math-reu/index.html>



## For Further Reading



[Diversity] with J. Maney, U. Krause,

Two preprints are available at:

<http://www-rohan.sdsu.edu/~vadim/research.html>



[WFD] D.D. Anderson, L.A. Mahaney,

On primary factorizations.

*J. Pure Appl. Algebra* 54 (2-3), 1988.



[WFM] F. Halter-Koch,

Divisor theories with primary elements and weakly Krull domains.

*Bull. Un. Mat. Ital. B*, 7(9)2, 1995.



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Semigroups that are factorial from inside or from outside.

*Lattices, Semigroups and Universal Algebra*, 1990.

