

A New Factorization Invariant

diversity in algebra

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<http://www-rohan.sdsu.edu/~vadim/diversity.pdf>



Motivation

Suppose $p|ab$, in some [semigroup/monoid/domain].

p *prime* implies $(p|a \text{ or } p|b)$. [UFM/UFD]

p *primary* implies $(p|a \text{ or } p|b \text{ or } (p \upharpoonright a \text{ and } p \upharpoonright b))$,

where $x \upharpoonright y$ means $x|y^n$ for some $n \in \mathbb{N}$. [WFM/WFD]

p *almost primary* implies $(p \upharpoonright a \text{ or } p \upharpoonright b)$. [?]

Ex: $2\mathbb{Z}$ 6 is almost primary, not primary: $6|2 \cdot 18$



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Almost-primary factorizations

A factorization into almost primary elements $p_1 p_2 \cdots p_k$ is *reduced* if $p_i \not\sim p_j$, for every p_i, p_j .

Note: If $p_i \mid p_j$, then $(p_i p_j)$ is almost primary.

Theorem 1

If x has any factorization into almost primary elements, then it has a reduced factorization into almost primary elements, unique up to order and weak associates ($p \mid q$ and $q \mid p$).

Theorem 2

Suppose the [monoid/domain] is atomic. All elements have such a unique factorization, if and only if all atoms are almost primary.



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The new invariant

Given [monoid/domain] X , with $x \in X$, we define the diversity of x as the largest cardinality of $S \subseteq X$, where $x \mid (\prod S)$, but $x \nmid (\prod \text{any proper subset of } S)$.
We write $\text{div}(x)$.

Properties:

$$\text{div}(xy) \leq \text{div}(x) + \text{div}(y)$$

$\text{div}(x) = 1$ if and only if x is almost primary (or a unit)

diversity \leq tame degree

diversity independent of catenary degree



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An example

$\mathbb{Z} \setminus \{0, -1\}$, under multiplication. 1 is the only unit.

Atoms are \pm rational primes.

Atoms almost primary, but not primary: $2 \mid (-2) \cdot 3$.

Atoms are *homogeneous*: For atom p , if $q \mid p$, then $p \mid q$.



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Please encourage your students to apply to the
San Diego State University Mathematics REU.

<http://www.sci.sdsu.edu/math-reu/index.html>



For Further Reading



[Diversity] with J. Maney, U. Krause,

Two preprints are available at:

<http://www-rohan.sdsu.edu/~vadim/research.html>



[WFD] D.D. Anderson, L.A. Mahaney,

On primary factorizations.

J. Pure Appl. Algebra 54 (2-3), 1988.



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Bull. Un. Mat. Ital. B, 7(9)2, 1995.



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Semigroups that are factorial from inside or from outside.

Lattices, Semigroups and Universal Algebra, 1990.

