The Probability that Two Semigroup Elements Commute Can Be Anything

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http://www-rohan.sdsu.edu/~vadim/commute.pdf

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Introduction

A finite semigroup S has an associative, closed, binary operation.

inverses, identity, commutativity are NOT assumed.

The commuting probability of S is the probability that $x \star y = y \star x$ if x, y are chosen uniformly at random.

Which probabilities in $(0, 1]$ are possible?

Previously: the set of possible values are dense in $(0, 1]$.

Now: the set of possible values are (all rationals in) $(0, 1]$.

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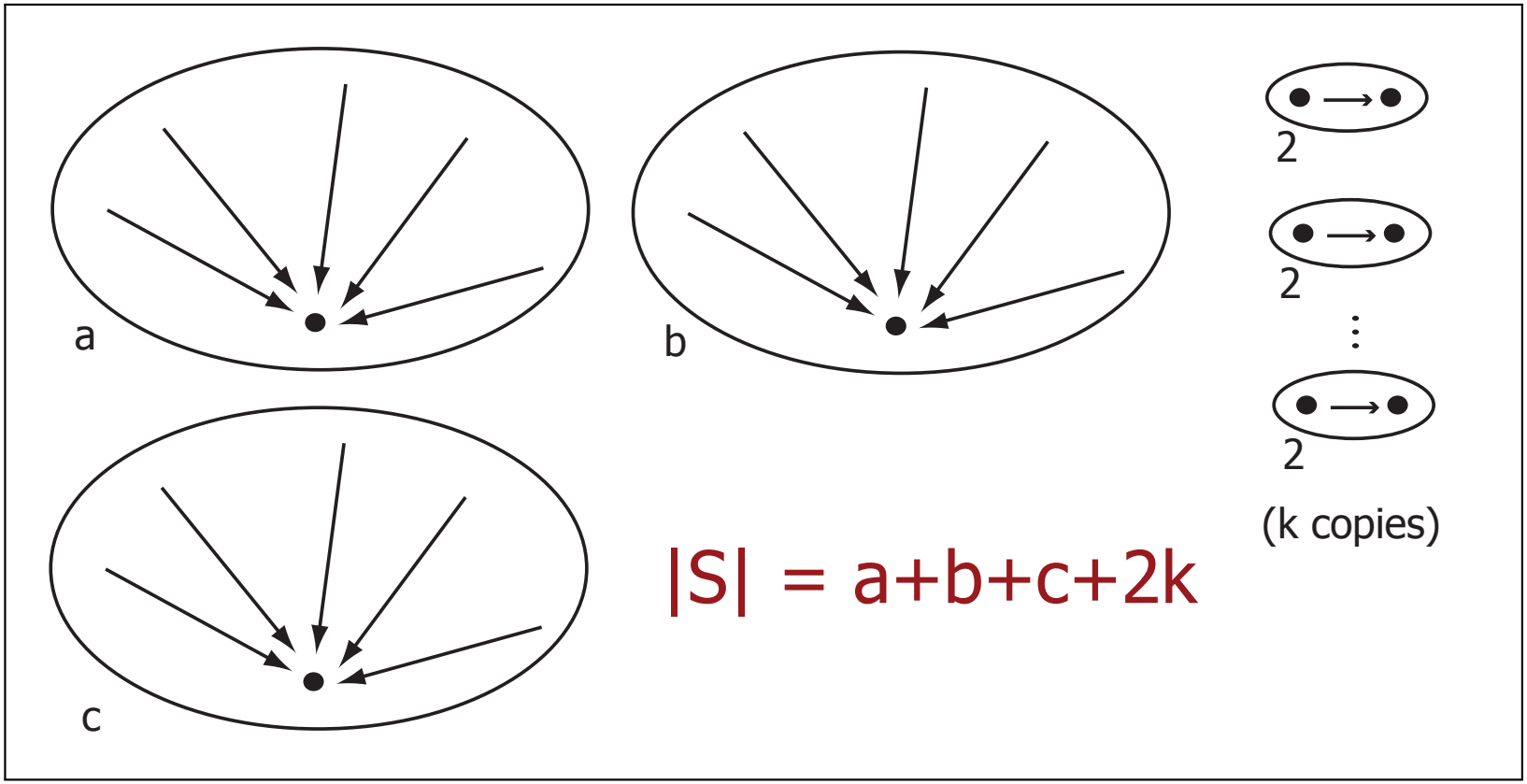
First Semigroup Family

Set $x \star y = f(x)$, where $f : S \to S$ is:



$$x \star y = f(x)$$

- f is idempotent, hence this is associative
- commuting probability is $\frac{a^2+b^2+c^2+4k}{(a+b+c+2k)^2}$.
- Using Lagrange's four-square theorem, can achieve every rational in $(0, 1/3]$.

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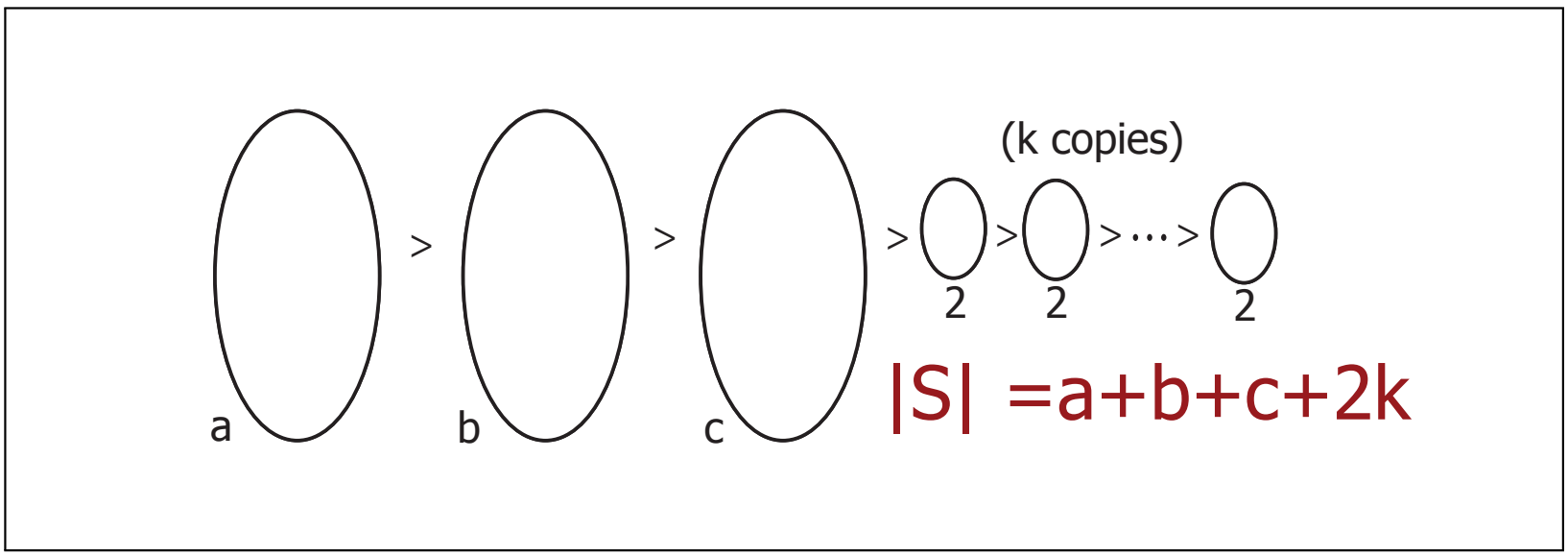
Second Semigroup Family

Rank function $r : S \to \mathbb{N}$ is given below.
If $r(x) > r(y)$ then $x \star y = y \star x = x$.
If $r(x) = r(y)$ then $x \star y = x$.



$x \star y = x$ or y

- $x \star y \star z$ is of those in $\{x, y, z\}$ with highest rank, the one that is first in the expression. Associative.
- commuting probability is $1 - \frac{a^2+b^2+c^2+4k-(a+b+c+2k)}{(a+b+c+2k)^2}$.
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First Semigroup Alteration

Given semigroup S, we make two copies S_1, S_2.

$x_1 \star y_2 = y_2 \star x_1 = x_1$; $\quad x_i \star y_i = (x \star y)_i$.

This is associative (several cases).

If S had commuting probability $\frac{m}{n}$, then this alteration has commuting probability $(\frac{m}{n} + 1)/2$.

Applying it to the first construction gives all rationals in $(1/2, 2/3]$.

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Second Semigroup Alteration

Given semigroup S, we make two copies S_1, S_2.

$x_i \star y_j = (x \star y)_i$.

$x_i \star y_j \star z_k = (x \star y \star z)_i$. Associative.

If S had commuting probability $\frac{m}{n}$, then this alteration has commuting probability $\frac{m}{2n}$.

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Conclusions

$$(0, 1/3] \cup (2/3, 1] \cup (1/2, 2/3] \cup (1/3, 1/2] = (0, 1].$$

Open: Find a single family.

Open: Use just two alterations with some trivial starter groups.

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