

LENGTH DENSITY AND NUMERICAL SEMIGROUPS

COLE BROWER, SCOTT CHAPMAN, TRAVIS KULHANEK, JOSEPH MCDONOUGH,
CHRISTOPHER O'NEILL, VODY PAVLYUK, AND VADIM PONOMARENKO

ABSTRACT. Length density is a recently introduced factorization invariant, assigned to each element n of a cancellative commutative atomic semigroup S , that measures how far the set of factorization lengths of n is from being a full interval. We examine length density of elements of numerical semigroups (that is, additive subsemigroups of the non-negative integers).

1. INTRODUCTION

A numerical semigroup S is an additively closed subset of $\mathbb{Z}_{\geq 0}$ containing 0, usually specified using a generating set n_1, \dots, n_k , i.e.,

$$S = \langle n_1, \dots, n_k \rangle = \{z_1 n_1 + z_2 n_2 + \dots + z_k n_k \mid z_1, \dots, z_k \in \mathbb{Z}_{\geq 0}\}.$$

A *factorization* of an element of S is an expression of the form $z_1 n_1 + \dots + z_k n_k$. Many classical problems surrounding factorizations in semigroup theory involve so-called factorization invariants, which are arithmetic quantities, often combinatorial in nature, that capture some precise aspect of non-uniqueness. For instance, the elasticity invariant equals the quotient of the maximum and minimum factorization lengths of an element $n \in S$, and the delta set invariant $\Delta_S(n)$ contains the “gaps” in the set of possible factorization lengths of n .

A recent paper [10] introduced a new invariant, known as *length density*, that measures the sparse-ness of the length set, defined in such a way that $\text{LD}_S(n) = 1$ if and only if the length set of n is a full interval. The length density of any numerical semigroup S has the following immediate upper and lower bounds [10, Proposition 2.3]:

$$\frac{1}{\max \Delta(S)} \leq \text{LD}(S) \leq \frac{1}{\min \Delta(S)}.$$

It is not hard to see that the second inequality above is strict precisely when $|\Delta(S)| > 1$. The strictness of the first inequality, on the other hand, turns out to be more nuanced. With this in mind, we introduce and study the following property: we say S is *tasty*¹ if $\text{LD}(S) > 1/\max \Delta(S)$, and *bland* otherwise. In order for S to be bland, some $n \in S$ must have $\Delta_S(n) = \{\max \Delta(S)\}$, a phenomenon that has been studied in the context of Krull monoids [8]. As such, determining when S is tasty necessitates sufficient control over the delta sets of elements of S .

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¹This term was chosen since the McNugget semigroup is tasty; see Example 3.1.

The aim of the present manuscript is to investigate how common the tasty and bland properties are among numerical semigroups. Our results are as follows. After reviewing the necessary background in Section 2, we provide in Section 3 an algorithm for computing $\text{LD}(S)$ for any numerical semigroup S , and characterize the asymptotic behavior of length density for large elements of S . In the remaining sections, we examine length density, and in particular the tastiness property, for several well-studied families of numerical semigroups. Several interesting consequences are also obtained.

- Any numerical semigroup with at most 3 generators has its length density achieved at a Betti element (Theorem 4.5). Note that a 4-generated numerical semigroup without this property is known (Example 4.6).
- If S has maximal embedding dimension and prime multiplicity, then the delta set of any Betti element containing $\max \Delta(S)$ must be a singleton. This resembles a property of certain Krull monoids [8] that generally does not hold for numerical semigroups.

2. BACKGROUND

We open with a series of definitions.

Definition 2.1. A *numerical semigroup* is a subset of $\mathbb{Z}_{\geq 0}$ of the form

$$S = \langle n_1, \dots, n_k \rangle = \{z_1 n_1 + \dots + z_k n_k : z_1, \dots, z_k \in \mathbb{Z}_{\geq 0}\}$$

for the semigroup generated by n_1, \dots, n_k . A *factorization* of $n \in S$ is an expression

$$n = z_1 n_1 + \dots + z_k n_k$$

of n as a sum of generators of S , and the *length* of a factorization is the sum $z_1 + \dots + z_k$. The *set of factorizations* of n is the set

$$\mathbf{Z}_S(n) = \{z \in \mathbb{Z}_{\geq 0}^k : n = z_1 n_1 + \dots + z_k n_k\}$$

viewed as a subset of $\mathbb{Z}_{\geq 0}^k$, and the *length set* of n is the set

$$\mathbf{L}_S(n) = \{z_1 + \dots + z_k : z \in \mathbf{Z}_S(n)\},$$

of all possible factorization lengths of n . Writing $\mathbf{L}_S(n) = \{\ell_1 < \dots < \ell_r\}$, define

$$\Delta_S(n) = \{\ell_i - \ell_{i-1} : 2 \leq i \leq r\} \quad \text{and} \quad \Delta(S) = \bigcup_{n \in S} \Delta_S(n)$$

as the *delta sets* of n and S , respectively.

Definition 2.2. Fix a numerical semigroup $S = \langle n_1, \dots, n_k \rangle$. The *factorization homomorphism* of S is the function $\varphi_S : \mathbb{Z}_{\geq 0}^k \rightarrow S$ given by

$$\varphi_S(z_1, \dots, z_k) = z_1 n_1 + \dots + z_k n_k$$

sending each k -tuple to the element of S it is a factorization of. The *kernel* of φ_S is the equivalence relation $\sim = \ker \varphi_S$ that sets $z \sim z'$ whenever $\varphi_S(z) = \varphi_S(z')$. The

kernel \sim is in fact a *congruence*, meaning that $z \sim z'$ implies $(z + z'') \sim (z' + z'')$ for all $z, z', z'' \in \mathbb{N}^k$. A subset $\rho \subset \ker \varphi_S$, viewed as a subset of $\mathbb{N}^k \times \mathbb{N}^k$, is a *presentation* of S if $\ker \varphi_S$ is the smallest congruence on \mathbb{N}^k containing ρ . We say ρ is a *minimal presentation* if it is minimal with respect to containment among presentations for S .

Definition 2.3. The *factorization graph* of an element n of a numerical semigroup S is the graph ∇_n whose vertices are the elements of $Z_S(n)$ in which two factorizations $z, z' \in Z_S(n)$ are connected by an edge if $z_i > 0$ and $z'_i > 0$ for some i . If ∇_n is disconnected, we say n is a *Betti element* of S , and we write $\text{Betti}(S)$ for the set of Betti elements of S .

The following theorem is a summary of multiple results which can be found in [1, Section 5.3].

Theorem 2.4. *A presentation ρ of a finitely generated semigroup $S = \langle n_0, \dots, n_k \rangle$ is minimal if and only if for every $n \in S$, (i) the number of connected components in the graph ∇_n is one more than the number of relations in ρ containing factorizations of n , and (ii) adding an edge to ∇_n corresponding to each such relation in ρ yields a connected graph.*

Our next definition yields our main property of interest.

Definition 2.5. Fix a numerical semigroup $S = \langle n_1, \dots, n_k \rangle$. The *length density* of $n \in S$ with $|\mathbf{L}_S(n)| \geq 2$ is given by

$$\text{LD}_S(n) = \frac{|\mathbf{L}(n)| - 1}{\max \mathbf{L}_S(n) - \min \mathbf{L}_S(n)}$$

and the *length density* of S is

$$\text{LD}(S) = \inf\{\text{LD}_S(n) : n \in S, |\mathbf{L}_S(n)| \geq 2\}.$$

It is not hard to prove (see [10]) that

$$\frac{1}{\max \Delta(S)} \leq \text{LD}(S) \leq \frac{1}{\min \Delta(S)}.$$

We say S is *tasty* if

$$\text{LD}(S) > \frac{1}{\max \Delta(S)}$$

and *bland* otherwise.

Before moving forward, we list a few basic results from [10] concerning the length density.

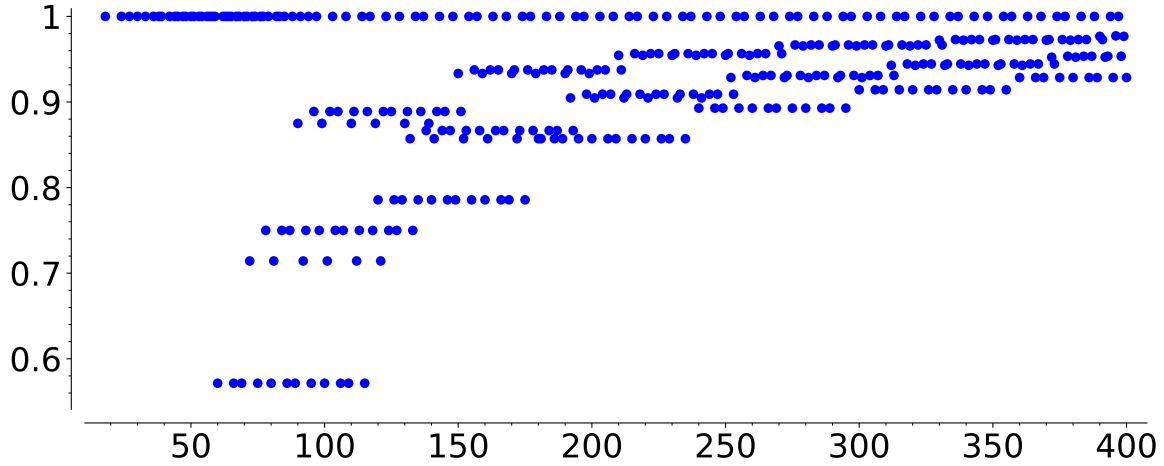


FIGURE 1. A plot with a point at $(n, \text{LD}_S(n))$ for each $n \in S = \langle 6, 9, 20 \rangle$.

Theorem 2.6 ([10, Theorem 3.4 and Proposition 3.2]). *For any numerical semigroup S , some $n \in S$ satisfies $\text{LD}_S(n) = \text{LD}(S)$. Additionally, S is bland if and only if*

$$\text{LD}_S(b) = \frac{1}{\max \Delta(S)}$$

for some $b \in \text{Betti}(S)$, which in particular occurs if $\Delta_S(b) = \{\max \Delta(S)\}$.

3. ASYMPTOTICS AND COMPUTATION

In this section, we characterize asymptotic behavior of $\text{LD}_S(n)$ for a given numerical semigroup S . Like many other factorization invariants [17, 18], for large n the function $\text{LD}_S(n)$ coincides with a quotient of quasilinear functions of n (Theorem 3.4). The primary consequence is an explicit upper bound on the smallest element of S with length density $\text{LD}(S)$ (Corollary 3.5), and thus an algorithm to compute $\text{LD}(S)$ (Remark 3.6).

Example 3.1. Figure 1 depicts the function $\text{LD}_S(n)$ for $S = \langle 6, 9, 20 \rangle$. Note in particular that S is tasty since $\text{LD}(S) = \frac{4}{7} > 1/\max \Delta(S) = \frac{1}{4}$.

Throughout this section, suppose $S = \langle n_1 < \dots < n_k \rangle$ is a numerical semigroup with $\gcd(n_1, \dots, n_k) = 1$, and let

$$d = \min \Delta(S) \quad \text{and} \quad L = \frac{n_k - n_1}{d \gcd(n_1, n_k)}.$$

Note $L \in \mathbb{Z}$ since $d = \gcd(n_2 - n_1, \dots, n_k - n_{k-1})$ by [5, Proposition 2.9].

We begin by recalling some pertinent results from [12], wherein the authors identify a constant N_S , given as a (large) formula in terms of n_1, \dots, n_k , such that

$$\Delta_S(n + \text{lcm}(n_1, n_k)) = \Delta_S(n).$$

for all $n \geq N_S$ [12, Corollary 14]. Letting

$$\begin{aligned}\mathbf{L}_1(n) &= \{\ell \in \mathbf{L}_S(n) \mid \frac{1}{n_1}n + N_S(\frac{1}{n_2} - \frac{1}{n_1}) < \ell \leq \frac{1}{n_1}n\}, \\ \mathbf{L}_2(n) &= \{\ell \in \mathbf{L}_S(n) \mid \frac{1}{n_k}n + N_S(\frac{1}{n_{k-1}} - \frac{1}{n_k}) \leq \ell \leq \frac{1}{n_1}n + N_S(\frac{1}{n_2} - \frac{1}{n_1})\}, \\ \mathbf{L}_3(n) &= \{\ell \in \mathbf{L}_S(n) \mid \frac{1}{n_k}n \leq \ell < \frac{1}{n_k}n + N_S(\frac{1}{n_{k-1}} - \frac{1}{n_k})\}\end{aligned}$$

so that

$$\mathbf{L}_S(n) = \mathbf{L}_1(n) \cup \mathbf{L}_2(n) \cup \mathbf{L}_3(n)$$

is a disjoint union, the following also follows directly from the work in [12].

Lemma 3.2. *If $n \geq N_S$, then*

$$\mathbf{L}_1(n + n_1) = \{\ell + 1 \mid \ell \in \mathbf{L}_1(n)\} \quad \text{and} \quad \mathbf{L}_3(n + n_k) = \{\ell + 1 \mid \ell \in \mathbf{L}_3(n)\},$$

and in particular,

$$|\mathbf{L}_1(n + n_1)| = |\mathbf{L}_1(n)|, \quad |\mathbf{L}_3(n + n_k)| = |\mathbf{L}_3(n)|, \quad \text{and} \quad \Delta(\mathbf{L}_2(n)) = \{d\}.$$

The following appeared as [17, Theorem 4.5] but without an explicit lower bound. We provide here a proof that N_S is such a lower bound.

Theorem 3.3. *If $n \geq N_S$, then*

$$|\mathbf{L}(n + \text{lcm}(n_1, n_k))| = |\mathbf{L}(n)| + L.$$

Proof. First, we set $p = \text{lcm}(n_1, n_r)$. Using Lemma 3.2, we have

$$\begin{aligned}|\mathbf{L}(n + p)| &= |\mathbf{L}_1(n + p)| + |\mathbf{L}_2(n + p)| + |\mathbf{L}_3(n + p)| \\ &= |\mathbf{L}_1(n)| + |\mathbf{L}_2(n + p)| + |\mathbf{L}_3(n)| \\ &= |\mathbf{L}(n)| + |\mathbf{L}_2(n + p)| - |\mathbf{L}_2(n)| \\ &= |\mathbf{L}(n)| + \frac{1}{d}(\max \mathbf{L}_2(n + p) - \min \mathbf{L}_2(n + p)) - \frac{1}{d}(\max \mathbf{L}_2(n) - \min \mathbf{L}_2(n)) \\ &= |\mathbf{L}(n)| + \frac{1}{d}(\min \mathbf{L}_1(n + p) - \max \mathbf{L}_3(n + p)) - \frac{1}{d}(\min \mathbf{L}_1(n) - \max \mathbf{L}_3(n)) \\ &= |\mathbf{L}(n)| + \frac{1}{d}(\min \mathbf{L}_1(n) - \max \mathbf{L}_3(n) + dL) - \frac{1}{d}(\min \mathbf{L}_1(n) - \max \mathbf{L}_3(n)) \\ &= |\mathbf{L}(n)| + L\end{aligned}$$

as desired. □

Theorem 3.4. *If $n \geq N_S$, then*

$$\text{LD}(n + \text{lcm}(n_1, n_k)) = \frac{|\mathbf{L}(n)| - 1 + L}{\max \mathbf{L}(n) - \min \mathbf{L}(n) + dL}.$$

In particular, $\text{LD}(n)$ is a quotient of eventually quasilinear functions of n , each with period dividing $\text{lcm}(n_1, n_k)$.

Proof. First, let $p = \text{lcm}(n_1, n_k)$. It follows from [4, Theorems 4.2 and 4.3] that

$$\max \mathbf{L}(n + n_1) = \max \mathbf{L}(n) + 1 \quad \text{and} \quad \min \mathbf{L}(n + n_k) = \min \mathbf{L}(n) + 1$$

hold for all $n \geq n_k^2$, and since $N_S > n_k^2$, one readily checks that

$$\begin{aligned} \text{LD}(n + p) &= \frac{|\mathbf{L}(s + p)| - 1}{\max(\mathbf{L}(s + p)) - \min(\mathbf{L}(s + p))} \\ &= \frac{|\mathbf{L}(s)| - 1 + L}{(\max(\mathbf{L}(s)) + \frac{1}{n_1}p) - (\min(\mathbf{L}(s)) + \frac{1}{n_k}p)} \\ &= \frac{|\mathbf{L}(s)| - 1 + L}{\max(\mathbf{L}(s)) - \min(\mathbf{L}(s)) + dL} \end{aligned}$$

holds for all $n \geq N_S$. □

Our final result of the section is a corollary of Theorem 3.4 that is topological in nature, reminiscent of [9, Theorems 2.1-2.2 and Corollary 2.3] and [4, Corollary 4.5] concerning the set of elasticities of a numerical semigroup.

Corollary 3.5. *We have*

$$\text{LD}(S) = \min\{\text{LD}(n) \mid n \in S \text{ and } n < N_S + \text{lcm}(n_1, n_k)\}.$$

Additionally, letting $R(S) = \{\text{LD}(n) : n \in S\}$, the set $R(S) \cap [0, \alpha]$ is finite for each $\alpha \in [0, \frac{1}{d}]$, and the only possible accumulation point of $R(S)$ is

$$\sup R(S) = \lim_{n \rightarrow \infty} \text{LD}(n) = \frac{1}{d}.$$

Proof. All 3 claims follow from Theorem 3.4 and the observation that

$$\text{LD}(n + \text{lcm}(n_1, n_k)) \geq \text{LD}(n)$$

for all $n \geq N_S$. □

Remark 3.6. Corollary 3.5 yield a method of computing $\text{LD}(S)$ from the generators of S . Indeed, N_S can be immediately computed from the formula in [12, Section 3], and the length sets of all $n \leq N_S + \text{lcm}(n_1, n_k)$ can be computed relatively quickly using the methods in [3, Section 3]. Additionally, Theorem 3.4 yields an algorithm to compute $\text{LD}_S(n)$ for $n \geq N_S$ whose runtime does not depend on n , since one only needs to compute the length set of an appropriate element between N_S and $N_S + \text{lcm}(n_1, n_k)$.

4. FAMILIES OF NUMERICAL SEMIGROUPS

In this section, we classify the length density and tastiness of numerical semigroups residing in one of several well-studied families.

4.1. Supersymmetric numerical semigroups. A numerical semigroup is called *supersymmetric* if it has exactly one Betti element. By [13, Theorem 12], this occurs if and only if S can be written in the form

$$S = \langle \frac{s}{t_1}, \frac{s}{t_2}, \dots, \frac{s}{t_k} \rangle$$

for some pairwise coprime $t_1 > \dots > t_k$ with $s = t_1 t_2 \cdots t_k$. Moreover, s is the unique Betti element in this case, and has length set $\mathbf{L}(s) = \{t_k, \dots, t_1\}$.

Lemma 4.1. *For any $m \geq 1$, $|\mathbf{L}(ms)| \geq mk - m + 1$.*

Proof. We proceed by induction on m . For the base case, notice that

$$|\mathbf{L}(s)| = |\{t_k, \dots, t_1\}| \geq (1)k - (1) + 1.$$

Next, assuming $|\mathbf{L}(ms)| \geq mk - m + 1$, we want to show

$$|\mathbf{L}((m+1)s)| \geq (m+1)k - (m+1) + 1.$$

Since $(m+1)s = ms + s$, it follows that

$$T := \mathbf{L}(ms) + t_k \subseteq \mathbf{L}((m+1)s)$$

has at least $mn - m + 1$ elements by our inductive hypothesis. Let ℓ denote the greatest element of $\mathbf{L}(ms)$, and fix a factorization (a_1, \dots, a_k) of ms of length ℓ . Then for each i , $(a_1, \dots, a_i + t_i, \dots, a_k)$ is a length $\ell + t_i$ factorization of $(m+1)s$. Since $\ell + t_k = \max T$,

$$\ell + t_1, \dots, \ell + t_{k-1} \in \mathbf{L}((m+1)s) \setminus T,$$

from which we conclude

$$|\mathbf{L}((m+1)s)| \geq (mk - m + 1) + k - 1 = (m+1)k - (m+1) + 1,$$

as desired. \square

Theorem 4.2. *If $n \in S$ has at least 2 factorizations, then $\text{LD}(n) \geq \text{LD}(s)$. Moreover,*

$$\text{LD}(S) = \text{LD}(s) = \frac{k-1}{t_1 - t_k},$$

and S is tasty if and only if t_1, t_2, \dots, t_k does not form an arithmetic progression.

Proof. By [13, Theorem 12], we can write $n = ms + r$ with $m \geq 1$ such that $r \in S$ has unique factorization, say with length ℓ . In this case, $\mathbf{L}(n) = \mathbf{L}(ms) + \ell$. Since $\max \mathbf{L}(ms) = mt_1$ and $\min \mathbf{L}(ms) = mt_n$, Lemma 4.1 yields

$$\text{LD}(n) = \text{LD}(ms) = \frac{|\mathbf{L}(ms)| - 1}{mt_1 - mt_k} \geq \frac{(mk - m + 1) - 1}{mt_1 - mt_k} = \frac{k-1}{t_1 - t_k} = \text{LD}(s).$$

The final claim now follows from the fact that $\Delta(s)$ is a singleton if and only if

$$\mathbf{L}(s) = \{t_k, \dots, t_1\}$$

forms an arithmetic progression. \square

4.2. Embedding dimension 3 numerical semigroups. Numerical semigroups with 3 generators have been well-studied, and it is known that their factorization structure is largely determined by whether there are 1, 2, or 3 Betti elements. Throughout this subsection, we use the notation from [19, Chapter 9], and refer the reader there for a thorough overview of this dichotomy.

Proposition 4.3. *If $S = \langle n_1, n_2, n_3 \rangle$ has 3 Betti elements, then S is bland.*

Proof. Each $b \in \text{Betti}(S)$ has exactly 2 factorizations, which, after relabeling the generators accordingly, have the form

$$b = c_1 n_1 = r_2 n_2 + r_3 n_3.$$

As such, either $\Delta(b) = \emptyset$ or $\Delta(b) = \{\delta\}$, where $\delta = |r_2 + r_3 - c_1|$. Thus, S is bland. \square

Proposition 4.4. *If $S = \langle n_1, n_2, n_3 \rangle$ with $\text{Betti}(S) = \{b_1, b_2\}$ and $b_1 < b_2$, then S is tasty if and only if $\max \Delta(b_2) > \max \Delta(b_1)$ and $b_2 - b_1 \in S$.*

Proof. For the backward direction, suppose $\max \Delta(b_2) > \max \Delta(b_1)$ and $b_2 - b_1 \in S$. We must have $\max \Delta(S) = \max \Delta(b_2)$ by [7, Theorem 2.5], but since $b_2 - b_1 \in S$, the trade defined at b_1 can be used at b_2 to obtain an element of $\Delta(b_2)$ at most $\max \Delta(b_1)$. As such, $|\Delta(b_2)| \geq 2$, so S is tasty.

Conversely, if $\max \Delta(b_2) \leq \max \Delta(b_1)$, then S is bland since $|\Delta(b_1)| = 1$, and if $b_2 - b_1 \notin S$, then b_1 and b_2 both have singleton delta sets, so again S is bland. This completes the proof. \square

Theorem 4.5. *If $S = \langle n_1, n_2, n_3 \rangle$, then $\text{LD}(S)$ occurs at a Betti element.*

Proof. If S has 1 Betti element, then S is supersymmetric, so apply Theorem 4.2. The claim clearly holds if S is bland, so by Propositions 4.3 and 4.4 it suffices to assume $\text{Betti}(S) = \{b_1, b_2\}$, $b_2 - b_1 \in S$, and $\max \Delta(b_2) > \max \Delta(b_1)$. Write $S = \langle n_1, n_2, n_3 \rangle$,

$$b_1 = c_1 n_1 = c_2 n_2 \quad \text{and} \quad b_2 = c_3 n_3 = r_1 n_1 + r_2 n_2,$$

and let

$$\delta_1 = c_1 - c_2 \quad \text{and} \quad \delta_2 = r_1 + r_2 - c_3.$$

Notice $t_1 = (c_1, -c_2, 0)$ and $t_2 = (r_1, r_2, -c_3)$ form a minimal presentation for S . Assume $c_1 > c_2$, and the two given factorizations of b_2 have extremal lengths in $\mathbf{L}(b_2)$ (or, equivalently, that $|\delta_2|$ is maximal among choices for the trade t_2).

Fix $n \in S$ not uniquely factorable. If $n - b_2 \notin S$, then only the trade t_1 is available, so $\Delta(n) = \{\delta_1\}$ and thus $\text{LD}(n) = \text{LD}(b_1)$. As such, suppose $n - b_2 \in S$. Fix a factorization $z = (z_1, z_2, z_3) \in \mathbf{Z}(n)$ with z_3 maximal, and write $z_3 = qc_3 + r$ with $q, r \in \mathbb{Z}_{\geq 0}$ and $r < c_3$. Since t_2 is the only trade involving n_3 , we must have

$$\{y_3 : (y_1, y_2, y_3) \in \mathbf{Z}(n)\} = \{r, r + c_3, \dots, r + qc_3\}.$$

Moreover, let $\ell = z_1 + z_2 + z_3 - c_3$, and let

$$L = \mathbf{L}(b_2) + \{\ell, \ell + \delta_2, \dots, \ell + (q - 1)\delta_2\}.$$

By repeatedly performing the trade t_2 to z , we see $L \subseteq \mathbf{L}(n)$, and by the maximality of $|\delta_2|$, L is a union of translations of $\mathbf{L}(b_2)$ with matching endpoints. In particular,

$$\text{LD}(L) = \text{LD}(b_2) > \text{LD}(b_1) = \frac{1}{\delta_1}.$$

The key observation is now that by the maximality of $|\delta_2|$, every length in $\mathbf{L}(n) \setminus L$ is attained by a factorization obtained from performing the trade t , perhaps multiple times, to some factorization whose length lies in L . As such, letting $m = |\mathbf{L}(n) \setminus L|$, we obtain

$$\text{LD}(n) \geq \frac{|L| + m}{\max L - \min L + m\delta_1} \geq \frac{|L|}{\max L - \min L} = \text{LD}(b_2),$$

as desired. \square

Example 4.6. The conclusion of Theorem 4.5 can fail if S has 4 or more generators. Indeed, the semigroup $S = \langle 20, 28, 42, 73 \rangle$, which appeared as [10, Example 3.3], has Betti element length sets $\mathbf{L}(84) = \{2, 3\}$, $\mathbf{L}(140) = \{4, 5, 7\}$, and $\mathbf{L}(146) = \{2, 4, 5\}$, but a strictly smaller length density results from $\mathbf{L}(202) = \{4, 6, 7, 9\}$.

4.3. Maximal embedding dimension numerical semigroups. We say S is *maximal embedding dimension* (or *MED*) if $e(S) = m(S)$. In this subsection, we write

$$S = \langle m, n_1, \dots, n_{m-1} \rangle$$

and assume $n_i \equiv i \pmod{m}$ for each i . It is known in this case that

$$\text{Betti}(S) = \{n_i + n_j : 1 \leq i \leq j \leq m - 1\};$$

we refer the reader to [19, Section 7.4].

We begin by identifying a class of Betti elements whose delta sets are singletons.

Proposition 4.7. *Suppose $b \in \text{Betti}(S)$ is the smallest Betti element in the equivalence class of $k \in [0, m - 1]$ modulo m . If $k = 0$ or k is a unit in \mathbb{Z}_m , then $|\mathbf{L}(b)| \leq 2$.*

Proof. First, suppose $k = 0$, so that b is the smallest Betti element of S divisible by m . We claim any factorization of b involving n_i with $1 \leq i \leq m - 1$ must have length 2 (and thus must be the factorization $n_i + n_{m-i}$). Indeed, by the minimality of b ,

$$n_i + n_{m-i} - b = cm$$

for some $c \geq 0$, so any factorization of b of length 3 or more involving n_i would yield a factorization of n_{m-i} of length at least 2, which is impossible since n_{m-i} is a minimal generator of S . This proves $\mathbf{L}(b) = \{2, a\}$, where $b = am$.

In all remaining cases, $b \equiv k \pmod{m}$ with $1 \leq k \leq m - 1$. Write

$$b = am + n_k = n_i + n_{k-i},$$

where $a \geq 2$ and $1 \leq i \leq m - 2$. By similar reasoning as above, any factorization involving n_i with $i \neq k$ must be the factorization $n_i + n_{k-i}$ by the minimality of b .

As such, since k is a unit in \mathbb{Z}_m , any factorization involving n_k other than $am + n_k$ must contain $(m+1)n_k$, which is impossible since

$$(m+1)n_k = cn_k + (m+1-c)n_k > n_i + n_{k-i} \geq b$$

for some c . This again yields the desired claim. \square

Theorem 4.8. *If $m \geq 3$ is prime, then S is bland.*

Proof. Fix $a \geq 1$ so that am is the smallest Betti element that is a multiple of m . By relabeling the generators of S using an appropriately chosen automorphism of \mathbb{Z}_m , it suffices to assume $am = n_1 + n_{m-1}$. In what follows, we show

$$\max \Delta(S) = a - 2,$$

which completes the proof since $\Delta(am) = \{a - 2\}$ by Proposition 4.7.

First, fix $b' = n_j + n_{m-j} \in \text{Betti}(S)$, and suppose $b' > am$. Let

$$b = n_i + n_{m-i} \in \text{Betti}(S)$$

in such a way that i is maximal such that $i < j$ and $b < b'$ (note $i = 1$ satisfies both constraints, so the set of eligible i is nonempty). By induction on b' , we can assume $\max \Delta(b) \leq a - 2$. Fix $c, c' \geq 1$ such that

$$b' = cm + n_i + n_{m-i} \quad \text{and} \quad n_{i+1} + n_{m-i-1} = c'm + n_i + n_{m-i}.$$

Since n_{i+1} is a minimal generator of S , $n_i + n_1 > n_{i+1}$ and thus $n_i + n_1 \geq n_{i+1} + m$. Similarly, $n_{m-i} + n_{m-1} \geq n_{m-i-1} + m$, and we obtain

$$c'm = n_{i+1} + n_{m-i-1} - (n_i + n_{m-i}) \leq n_1 + n_{m-1} - 2m = m(a - 2),$$

so by maximality of i , we have $c \leq c' \leq a - 2$. As such, from

$$\{2\} \cup (c + 2 + \mathbf{L}(b)) \subseteq \mathbf{L}(b'),$$

we conclude $\max \Delta(b') \leq a - 2$.

Next, fix $k \in [1, m - 1]$. Letting $b = n_1 + n_{k-1} = cm + n_k$, for some c , we see

$$cm = n_1 + n_{k-1} - n_k \leq n_1 + n_{m-1} - m = m(a - 1).$$

This means $\mathbf{L}(b) \subset [2, c + 1] \subset [2, a]$, so $\max \Delta(b) \leq a - 2$. More generally, let $b' = n_j + n_{k-j}$ with $1 \leq j \leq k - j \leq k$, and let $b = n_i + n_{j-i}$ with i maximal subject to $i < j$ and $b < b'$. By induction on b' , we can assume $\max \Delta(b) \leq a - 2$. Fix c, c' so that

$$b' = cm + n_i + n_{k-i} \quad \text{and} \quad n_{i+1} + n_{k-i-1} = c'm + n_i + n_{k-i}.$$

We obtain

$$c'm = n_{i+1} + n_{k-i-1} - (n_i + n_{k-i}) \leq n_1 + n_{m-1} - 2m = m(a - 2),$$

so by the maximality of i , we have $c \leq c' \leq a - 2$. As before, we have

$$\{2\} \cup (c + 2 + \mathbf{L}(b)) \subseteq \mathbf{L}(b'),$$

and can conclude $\max \Delta(b') \leq a - 2$. A similar argument for $b' = n_{k+j} + n_{m-j}$ with $k + 1 \leq k + j \leq m - j \leq m - 1$ completes the proof. \square

Proposition 4.9. *There exist both bland and tasty MED numerical semigroups of each composite multiplicity.*

Proof. The numerical semigroup $S = \langle m, m + 1, \dots, 2m - 1 \rangle$ is bland since it is arithmetical and thus has singleton delta set by [2, Theorem 2.2].

Now, if $m = pq$ with p prime and $q \geq 2$, then we claim $S = \langle m, n_1, \dots, n_{m-1} \rangle$ with

$$n_i = \begin{cases} m + i & \text{if } p \mid i; \\ 2qm + i & \text{if } p \nmid i, \end{cases}$$

is tasty. Indeed, consider a Betti element $b = n_i + n_j$. For convenience in what follows, we write $n_{i+j} = n_{i+j-m}$ if $i + j > m$. If $p \mid i$, then $p \mid j$ if and only if $p \mid (i + j)$, so

$$n_i + n_j = \begin{cases} m + n_{i+j} & \text{if } i + j < m; \\ 2m + n_{i+j-m} & \text{if } i + j > m, \end{cases}$$

and thus $L(b) \subseteq \{2, 3\}$. Alternatively, if $p \nmid i$ and $p \nmid j$, then $b = n_{i+j} + cm$ with $c \geq 2q$, so the trade $qn_p = (q + 1)m$ can be performed at least once, meaning $\min \Delta(b) = 1$.

The above argument implies each $b \in \text{Betti}(S)$ has either $\Delta(b) = \emptyset$ or $1 \in \Delta(b)$. Lastly, the Betti element $n_1 + n_{m-1} = (2B + 1)m$ has no factorizations of length 3, since any such factorization can use only m, n_p, \dots, n_{m-p} but $3n_{m-p} < n_1 + n_{m-1}$. Thus, $\max \Delta(S) > 1$ and occurs at a Betti element with non-singleton delta set. \square

We close this section by examining multiplicity 4 MED numerical semigroups, where geometry plays a role in determining whether each semigroup is bland or tasty.

Given $n_1, n_2, n_3 > 4$ with $n_i \equiv i \pmod{4}$ for each i , the semigroup $S = \langle 4, n_1, n_2, n_3 \rangle$ is MED if and only if

$$2n_1 > n_2 \quad n_1 + n_2 > n_3, \quad n_2 + n_3 > n_1, \quad \text{and} \quad 2n_3 > n_2$$

each hold [15]. As such, it is natural to representing each MED numerical semigroup $S = \langle 4, n_1, n_2, n_3 \rangle$ as a point $(n_1, n_2, n_3) \in \mathbb{R}^3$. Examining semigroups with fixed coordinate sum $n_1 + n_2 + n_3$ yields a cross section as depicted on the left in Figure 2. The two regions labeled “bland” coincide with the semigroups where $\min(n_1, n_2, n_3)$ equals n_1 and n_3 , respectively (Proposition 4.10). For the remaining semigroups, if $n_1 + n_3$ is sufficiently larger than n_2 , then S is guaranteed tasty by (Theorem 4.12) and lies in the region labeled “tasty” on the left. This phenomenon is also visible in the plot on the right in Figure 2, which depicts the cross section $n_2 = 18$, placing a large (red) point at (n_1, n_3) if S is bland and a smaller (black) point if S is tasty.

We note that the geometric viewpoint discussed above, first outlined in [15], has proven fruitful in recent years for studying enumerative questions surrounding numerical semigroups; we direct the interested reader to [6, 14].

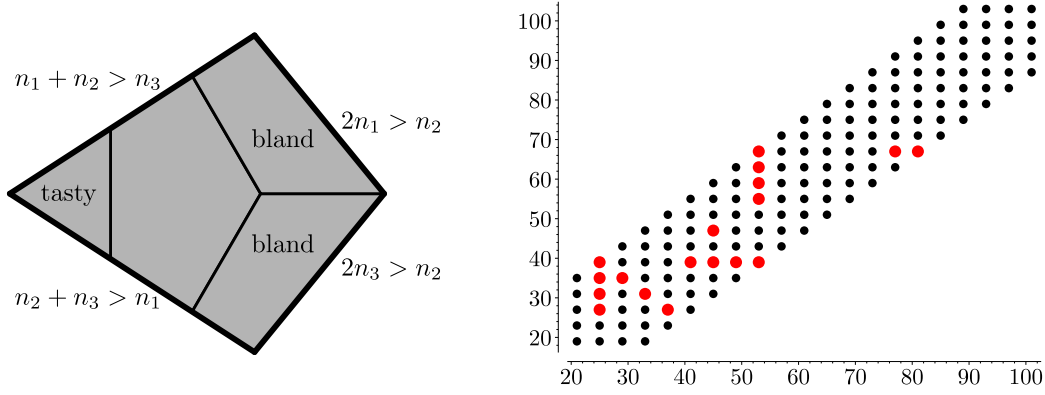


FIGURE 2. A diagram (left) of a cross section of MED numerical semi-groups $S = \langle 4, n_1, n_2, n_3 \rangle$ with $n_1 + n_2 + n_3$ fixed, and a plot (right) obtained by setting $n_2 = 18$ and placing a large (red) point at (n_1, n_3) if S is bland and a smaller (black) point if S is tasty.

Proposition 4.10. *If $m = 4$ and $\min(n_1, n_2, n_3) \neq n_2$, then S is bland.*

Proof. Throughout this proof, we assume $n_1 = \min(n_1, n_2, n_3)$, as an analogous argument follows upon reversing the roles of n_1 and n_3 throughout.

The Betti elements $n_1 + n_2$ and $n_2 + n_3$ are unique in their respective equivalence classes, and have singleton delta set by Proposition 4.7. Consider the Betti elements $b = 4a$ and $b' = 4(a + c)$ with $a, c \geq 0$. Again by Proposition 4.7, b has singleton delta set, and for b' , there are two cases. First, suppose

$$b = 4a = n_1 + n_3 \quad \text{and} \quad b' = 4(a + c) = 4c + n_1 + n_3 = 2n_2.$$

Clearly $L(b') \subseteq [2, a + c]$, so $\max \Delta(b') \leq \max(c, a - 2)$. We know $\Delta(b) = \{a - 2\}$, and since n_2 is a minimal generator of S ,

$$2n_1 \geq n_2 + 4 \quad \text{and} \quad 2n_3 \geq n_2 + 4,$$

so we obtain

$$4c = 2n_2 - (n_1 + n_3) \leq n_1 + n_3 - 8 = 4(a - 2)$$

and thus $c \leq a - 2$. As such, $\max \Delta(b') \leq \max \Delta(b)$. Second, suppose

$$b = 4a = 2n_2 \quad \text{and} \quad b' = 4(a + c) = 4c + 2n_2 = n_1 + n_3.$$

Again, clearly $L(b') \subseteq [2, a + c]$. Since n_2 and n_3 are minimal generators of S , we have

$$2n_1 \geq n_2 + 4 \quad \text{and} \quad n_1 + n_2 \geq n_3 + 4,$$

from which we obtain $3n_1 \geq n_3 + 8$. From there, $n_1 < n_2$ implies

$$4c = n_1 + n_3 - 2n_2 \leq 4n_1 - 2n_2 - 8 \leq 2n_2 - 8 = 4(a - 2)$$

and we again conclude $c \leq a - 2$ and $\max \Delta(b') \leq \max \Delta(b)$.

This leaves the Betti elements $2n_1$ and $2n_3$. Write

$$2n_1 = n_2 + 4d \quad \text{and} \quad 2n_3 = 2n_1 + 4e$$

for $d, e \geq 1$. Now, since $n_1 < n_2$, we have $4d = 2n_1 - n_2 < n_1 < 4a$, so $d \leq a - 1$. As such, since $\mathsf{L}(2n_1) \subset [2, d + 1]$, we have

$$\max \Delta(2n_1) \leq a - 2 = \max \Delta(b).$$

Lastly, from $n_1 + n_2 \geq n_3 + 4$ and $3n_1 \geq n_3 + 8$, we obtain

$$4e = 2n_3 - 2n_1 \leq 2n_2 - 8 \quad \text{and} \quad 4e = 2n_3 - 2n_1 \leq n_1 + n_3 - 8,$$

so $e \leq a - 2$ and thus $\max \Delta(2n_3) \leq a - 2$ as well. \square

By examining where in the proof of Proposition 4.10 the hypothesis $n_2 < n_1$ is used, we obtain the following.

Corollary 4.11. *If $m = 4$, then $\max \Delta(S)$ occurs at $\max(2n_2, n_1 + n_3)$ or $\min(2n_1, 2n_3)$.*

Proof. By the proof of Proposition 4.10, it suffices to ensure $\max \Delta(S)$ does not occur at

$$b = n_1 + n_2 = 4d + n_3 \quad \text{or} \quad b' = n_2 + n_3 = 4e + n_1.$$

Writing $4a = \min(2n_2, n_1 + n_3)$, summing these equalities above yields

$$4d + 4e = 2n_2 \geq 4a,$$

so $\max \Delta(b) = d - 1 \leq a - 2$ and $\max \Delta(b') = e - 1 \leq a - 2$. \square

Theorem 4.12. *Suppose $m = 4$ and $n_2 < n_1, n_3$. If $2n_1 + 2n_3 > n_2^2$, then S is tasty. In particular, for fixed n_2 , there are only finitely many bland numerical semigroups.*

Proof. As in the proof of Proposition 4.10, let $2n_2 = 4a$ so that $\mathsf{L}(2n_2) = \{2, a\}$. Let

$$b = n_1 + n_3 = 4c$$

for $c \geq 1$, and write $c = qa + r$ with $0 \leq r < a$. Any factorization of b aside from $n_1 + n_3$ uses only n_2 and 4, the shortest of which is

$$4r + 2qn_2 = 4r + 4aq = 4c = b.$$

This means $\Delta(b) = \{a - 2, 2q + r - 2\}$, so S is tasty if $2q + r \neq a$. This is achieved by

$$4a(2q + r) = 8qa + 4ar > 8qa + 8r = 8c = 2n_1 + 2n_3 > n_2^2 = 4a^2$$

whenever $2n_1 + 2n_3 > n_2^2$.

Supposing $n_1 < n_3$, by Corollary 4.11 it remains to show

$$b' = 2n_1 = 4d + n_2$$

is either non-singleton or does not contain $\max \Delta(S)$. Since $n_1 + n_2 \geq n_3 + 4$, we have

$$8d = 4n_1 - 2n_2 \geq 2n_1 + 2n_3 > n_2^2 > 4n_2 = 8a,$$

meaning some element of $\Delta(b')$ is at most $a - 2$, as desired. An analogous argument when $n_3 < n_1$ completes the proof. \square

5. TASTY AND BLAND GLUINGS OF NUMERICAL SEMIGROUPS

Given two numerical semigroups $S_1 = \langle n_1, \dots, n_r \rangle$ and $S_2 = \langle n_{r+1}, \dots, n_k \rangle$ and non-atoms $\lambda \in S_1$ and $\mu \in S_2$ such that $\gcd(\lambda, \mu) = 1$, we say the numerical semigroup

$$\mu S_1 + \lambda S_2 = \langle \mu n_1, \dots, \mu n_r, \lambda n_{r+1}, \dots, \lambda n_k \rangle$$

is a *gluing* of S_1 and S_2 by λ and μ . It is known that the above generating set for S is minimal, and that

$$\text{Betti}(S) = \mu \text{Betti}(S_1) \cup \lambda \text{Betti}(S_2) \cup \{\lambda\mu\}.$$

For more background on gluings, see [19, Chapter 8].

In this section, we investigate the following question.

If S_1 and S_2 are fixed and μ and λ are allowed to vary, what can be said about how often the gluing $\mu S_1 + \lambda S_2$ is tasty, and how often it is bland?

Our results are extremal in nature. Theorem 5.1 implies there will always be infinitely many tasty gluings $\mu S_1 + \lambda S_2$ for fixed S_1 and S_2 , but the same does not hold for bland gluings. In particular, it is possible for two given numerical semigroups S_1 and S_2 to have infinitely many bland gluings (Theorem 5.2), finitely many bland gluings (Theorem 5.3), or no bland gluings (Theorem 5.5).

Theorem 5.1. *Given numerical semigroups S_1 and S_2 , there are infinitely many gluings $S = \mu S_1 + \lambda S_2$ that are tasty.*

Proof. Let n_k denote the largest atom of S_2 , and let $d = \max(\Delta(S_1) \cup \Delta(S_2))$. First, choose any prime $\lambda \in S_1$ such that $|\mathbf{L}_{S_1}(\lambda)| \geq 2$ and $\lambda > n_2$. Next, choose any prime $p > \lambda$ such that

$$p \geq \max \mathbf{L}_{S_1}(\lambda) + d + 1$$

and let $\mu = pn_k$ so that $\min \mathbf{L}_{S_2}(\mu) = p$. Clearly $\gcd(\lambda, \mu) = 1$. Letting $S = \mu S_1 + \lambda S_2$, by [19, Theorem 8.2] we have

$$\mathbf{L}_S(\lambda\mu) = \mathbf{L}_{S_1}(\lambda) \cup \mathbf{L}_{S_2}(\mu)$$

and thus

$$\max \Delta(S) = \max \Delta_S(\lambda\mu) = \min \mathbf{L}_{S_2}(\mu) - \max \mathbf{L}_{S_1}(\lambda) = p - \max \mathbf{L}_{S_1}(\lambda) \geq d + 1,$$

which ensures S is tasty since $|\mathbf{L}_{S_1}(\lambda)| \geq 2$. □

Theorem 5.2. *Fix numerical semigroups S_1 and S_2 . If S_1 is bland and*

$$\max \Delta(S_1) \geq \max \Delta(S_2),$$

then there exists infinitely many gluings $\mu S_1 + \lambda S_2$ that are bland.

Proof. Let n_k denote the largest atom of S_2 . Choose any prime $\lambda \in S_1$ such that

$$\lambda > \max \text{Betti}(S_1), \quad \lambda > n_2, \quad \text{and} \quad \max \mathbf{L}_{S_1}(\lambda) > \max \text{Betti}(S_2),$$

and let $\mu = (\max \mathbf{L}_{S_1}(\lambda))n_k$ so that $\min \mathbf{L}_{S_2}(\mu) = \max \mathbf{L}_{S_1}(\lambda)$. Clearly $\max \mathbf{L}_{S_1}(\lambda) < \lambda$, so $\gcd(\lambda, \mu) = 1$. Letting $S = \mu S_1 + \lambda S_2$, we see $\max \text{Betti}(S) = \lambda\mu$, ensuring

- $\mathbf{L}_S(\mu b) = \mathbf{L}_{S_1}(b)$ for each $b \in \text{Betti}(S_1)$,
- $\mathbf{L}_S(\lambda b) = \mathbf{L}_{S_2}(b)$ for each $b \in \text{Betti}(S_2)$, and
- $\mathbf{L}_S(\lambda\mu) = \mathbf{L}_{S_1}(\lambda) \cup \mathbf{L}_{S_2}(\mu)$ with $\Delta_S(\lambda\mu) \leq \max(\Delta(S_1) \cup \Delta(S_2))$.

We conclude $\max \Delta(S)$ is attained at a Betti element with singleton delta set. \square

Theorem 5.3. *Let $S_1 = \langle 2, 3 \rangle$ and $S_2 = \langle 6, 9, 20 \rangle$. A gluing $S = \mu S_1 + \lambda S_2$ is bland if and only if $\lambda = 4$ and $\mu = 27$.*

Proof. Note that $\text{Betti}(S) = \{6\mu, 18\lambda, 60\lambda, \lambda\mu\}$, and that $\Delta_S(6\mu)$, $\Delta_S(18\lambda)$, and $\Delta_S(60\lambda)$ each contain 1.

First, suppose $\mu > 61$. In this case, $\mu\lambda > 60\lambda > 18\lambda$, so $\max \Delta(S) \geq \max \Delta(S_2) = 4$, and $1 \in \Delta_S(\lambda\mu)$ since $1 \in \Delta_{S_2}(\mu)$, so S must be tasty.

Next, suppose $\mu \leq 60$ and $\lambda > 33$. In this case, $1 \in \Delta_{S_1}(\lambda)$ and thus $1 \in \Delta_S(\lambda\mu)$, so S is tasty if and only if $\max \Delta(S) > 1$. We have $\mathbf{L}_{S_2}(\mu) \subseteq [2, 10] \cap \mathbb{Z}$, and since $\lambda > 33$, $\min \mathbf{L}_{S_1}(\lambda) \geq 12$. From this, we conclude $\max \Delta_S(\lambda\mu) \geq 2$ and thus S is tasty.

At this point, only finitely many gluings remain, and an exhaustive computation with the GAP package `numericalsgps` [11] completes the proof. \square

Example 5.4. We briefly examine the exceptional case identified in Theorem 5.3. Let $S_1 = \langle 2, 3 \rangle$, $S_2 = \langle 6, 9, 20 \rangle$, $\lambda = 4$, and $\mu = 27$, and let

$$S = \mu S_1 + \lambda S_2 = \langle 2\mu, 3\mu, 6\lambda, 9\lambda, 20\lambda \rangle.$$

From $\lambda\mu = 6\mu + 6\lambda + 2\mu$, one can readily verify

$$\mathbf{L}_S(60\lambda) = \mathbf{L}_{S_1}(\lambda) \cup (\mathbf{L}_S(6\mu) + 2) = \{3, 7, 8, 9, 10\} \cup \{4, 5, 6, 7\}$$

and

$$\mathbf{L}_S(\lambda\mu) = \mathbf{L}_{S_1}(\lambda) \cup \mathbf{L}_{S_2}(\mu) = \{2, 4\} \cup \{3\},$$

resulting in $\Delta(S) = \{1\}$ and ensuring S is bland.

Theorem 5.5. *Every gluing $S = \mu S_1 + \lambda S_2$ of $S_1 = \langle 2, 3 \rangle$ and $S_2 = \langle 6, 9, 26 \rangle$ is tasty.*

Proof. We begin by noting $\text{Betti}(S_2) = \{18, 78\}$, $\Delta_{S_2}(18) = \{1\}$, and $\Delta_{S_2}(78) = \{1, 6\}$, and that every $n \in S_2$ with $n > 73$ has $|\mathbf{L}_{S_2}(n)| \geq 2$.

First, suppose $\mu > 78$. It is easy to show that 1 lies in $\Delta_S(b)$ for each $b \in \text{Betti}(S)$. Since $1 \in \Delta_{S_2}(\mu)$ implies $1 \in \Delta_S(\mu\lambda)$, and since $\lambda\mu$ is the largest Betti element, 1 lies in the delta set of each $b \in \text{Betti}(S)$. Furthermore, $\max \Delta_S(78\lambda) = 6$, so S is tasty.

Next, suppose $\mu \leq 78$ and $\lambda > 42$. We have $\min \mathbf{L}_{S_1}(\lambda) \geq 15$ and $\mathbf{L}_{S_2}(\mu) \subseteq [2, 13] \cap \mathbb{Z}$, so $\max \Delta_S(\lambda\mu) \geq 2$, and $1 \in \Delta_S(\lambda\mu)$ since $1 \in \Delta_{S_1}(\mu)$. This again implies S is tasty.

Now, all remaining gluings satisfy $\lambda \leq 42$ and $\mu \leq 78$, and an exhaustive computation using [11] verifies each is tasty. \square

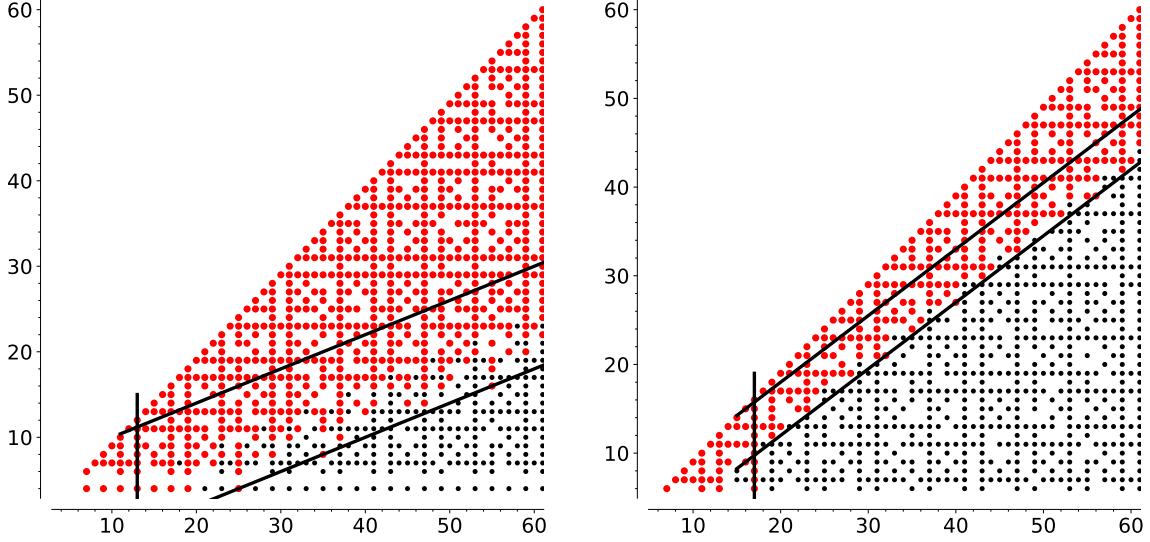


FIGURE 3. Plots for $S = \langle 2, 5 \rangle$ (left) and $S = \langle 3, 4 \rangle$ (right) with a large (red) dot at (λ, μ) if $\mu S + \lambda S$ is tasty, and a small (black) dot if $\mu S + \lambda S$ is bland. Each linear constraint in Proposition 5.6 is also depicted.

Our final result of this section concerns gluings $\mu S + \lambda S$ of a numerical semigroup S with itself. In the case $e(S) = 2$, we identify the asymptotic proportion of gluings $\mu S + \lambda S$ that are tasty (Theorem 5.7). The plots in Figure 3 depicts the choices of λ and μ that are assured tasty or bland by Proposition 5.6.

Proposition 5.6. Fix $S = \langle n_1, n_2 \rangle$ and integers $\lambda > \mu > 2n_1n_2 - n_1 - n_2$ with $\gcd(\lambda, \mu) = 1$, and denote the gluing $G = \mu S + \lambda S$.

- (a) If $\lambda > \frac{n_2}{n_1}\mu + n_2(n_2 - n_1)$, then G is tasty.
- (b) If $\lambda < \frac{n_2}{n_1}\mu - n_2(n_2 - n_1)$, then G is bland.

Proof. The lower bound on λ and μ ensures $|\mathsf{L}_S(\lambda)| \geq 2$ and $|\mathsf{L}_S(\mu)| \geq 2$, and that $\lambda\mu$ is the largest Betti element of G . If $\lambda > \frac{n_2}{n_1}\mu + n_2(n_2 - n_1)$, then

$$\min \mathsf{L}_S(\lambda) \geq \frac{1}{n_2}\lambda > \frac{1}{n_1}\mu + (n_2 - n_1) \geq \max \mathsf{L}_S(\mu) + \max \Delta(S),$$

and thus $\max \Delta(G) = \max \Delta_G(\lambda\mu) > \max \Delta(S)$. From this, we conclude G is tasty. If, on the other hand, $\lambda < \frac{n_2}{n_1}\mu - n_2(n_2 - n_1)$, then

$$\min \mathsf{L}_S(\lambda) - \max \mathsf{L}_S(\mu) < \left(\frac{1}{n_2}\lambda + (n_2 - n_1)\right) - \left(\frac{1}{n_1}\mu - (n_2 - n_1)\right) < n_2 - n_1,$$

and similarly

$$\max \mathsf{L}_S(\lambda) - \min \mathsf{L}_S(\mu) > \frac{1}{n_1}\lambda - \frac{1}{n_2}\mu - 2(n_2 - n_1) > 0.$$

From these inequalities, we conclude there exists $\delta \in \Delta_G(\lambda\mu)$ with $0 < \delta < n_2 - n_1$, and since $\mathsf{L}_S(\lambda\mu) = \mathsf{L}_{S_1}(\lambda) \cup \mathsf{L}_{S_2}(\mu)$, we must have $\Delta_G(\lambda\mu) = \{\delta, n_2 - n_1\}$. It follows that $\max \Delta(G) = n_2 - n_1$, so $\Delta_G(\lambda n_1 n_2) = \{n_2 - n_1\}$ implies G is bland. \square

Theorem 5.7. *Given a numerical semigroup $S = \langle n_1, n_2 \rangle$, the proportion of tasty gluings of S with itself is n_1/n_2 . More precisely,*

$$\lim_{N \rightarrow \infty} \frac{\# \text{ tasty gluings } \mu S + \lambda S \text{ with } \lambda, \mu < N}{\# \text{ gluings } \mu S + \lambda S \text{ with } \lambda, \mu < N} = \frac{n_1}{n_2}.$$

Proof. Let L denote the limit on the left hand side above. Writing

$$A = \{(a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1\}$$

and $N_0 = 2n_1n_2 - n_1 - n_2$, Proposition 5.6(a) yields a lower bound on L of

$$\begin{aligned} L &\geq \lim_{N \rightarrow \infty} \frac{\# (\lambda, \mu) \in A \cap [N_0, N]^2 \text{ with } \lambda > \mu \text{ and } \lambda > \frac{n_2}{n_1}\mu + n_2(n_2 - n_1)}{\# (\lambda, \mu) \in A \cap [2, N]^2 \text{ with } \lambda > \mu} \\ &= \lim_{N \rightarrow \infty} \frac{\# (\lambda, \mu) \in [2, N]^2 \text{ with } \lambda > \mu \text{ and } \lambda > \frac{n_2}{n_1}\mu + n_2(n_2 - n_1)}{\# (\lambda, \mu) \in [2, N]^2 \text{ with } \lambda > \mu} \\ &= \lim_{N \rightarrow \infty} \frac{\# (\lambda, \mu) \in [2, N]^2 \text{ with } \frac{n_1}{n_2}\lambda > \mu}{\# (\lambda, \mu) \in [2, N]^2 \text{ with } \lambda > \mu} = \frac{n_1}{n_2} \end{aligned}$$

where the first equality follows from [16, Chapter IV, Theorem 1]. Proposition 5.6(b) yields an identical upper bound for L , so we conclude $L = n_1/n_2$. \square

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MATHEMATICS DEPARTMENT, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CA 92182
Email address: cole.brower@gmail.com

DEPARTMENT OF MATHEMATICS AND STATISTICS, SAM HOUSTON STATE UNIVERSITY, HUNTSVILLE, TX 77341
Email address: scott.chapman@shsu.edu

MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA LOS ANGELES, LOS ANGELES, CA 90095
Email address: volcanek47@gmail.com

MATHEMATICS DEPARTMENT, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CA 92182
Email address: joe.mcdonough@live.com

MATHEMATICS DEPARTMENT, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CA 92182
Email address: cdoneill@sdsu.edu

MATHEMATICS DEPARTMENT, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CA 92182
Email address: vody.pa@gmail.com

MATHEMATICS DEPARTMENT, SAN DIEGO STATE UNIVERSITY, SAN DIEGO, CA 92182
Email address: vponomarenko@sdsu.edu