Skeleton Calculus

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I. Overview

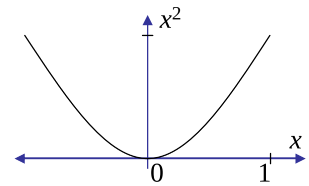
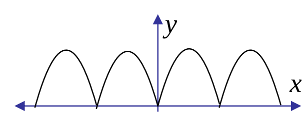
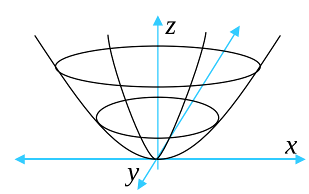
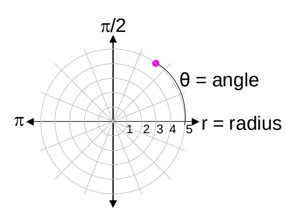
Basic calculus relies on 4 major concepts:

1. Functions
2. Limits
3. Derivatives
4. Integrals

Functions

A **function** takes one or more real values as inputs, and produces one or more real values as outputs. The inputs to a function are called the **arguments**. The simplest case is a real-valued function of a real-valued argument ($f : R^1 \rightarrow R^1$), e.g., $f(x) = \sin x$. A function which produces more than one output may be considered a vector-valued function.

There are 4 cases of interest:

Case	Example	
1. $f : R^1 \rightarrow R^1$	$y = f(x) = x^2$	
2. $\vec{f} : R^1 \rightarrow R^2$	$(x, y) = \vec{f}(t) = (t + \cos t, \sin t)$	
3. $f : R^2 \rightarrow R^1$	$z = f(x, y) = x^2 + y^2$	
4. $\vec{f} : R^2 \rightarrow R^2$	$(r, \theta) = \vec{f}(x, y) = \left(\sqrt{x^2 + y^2},\ \arctan \dfrac{y}{x} \right)$	

II. Limits

A. Definition (for a real-valued function of a single argument, $f : R^1 \to R^1$):

L is the **limit** of $f(x)$ as x approaches a, iff for every $\varepsilon > 0$, there exists a δ (> 0) such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In symbols:

$$L = \lim_{x \to a} f(x) \text{ iff } \forall \varepsilon > 0, \exists\, \delta \ni \left| f(x) - L \right| < \varepsilon \text{ whenever } 0 < \left| x - a \right| < \delta$$

This says that the value of the function *at a* doesn't matter; in fact, most often the function is not defined at a. However, the behavior of the function *near a* is important. If you can make the function arbitrarily close to some number, L, by restricting the function's argument to a small neighborhood around a, then L is the limit of f as x approaches a.

Example: Show that $\lim_{x \to 1} \frac{2x^2 - 2}{x - 1} = 4$. We prove the existence of δ given any ε by computing the necessary δ from ε. Note that for $x \neq 1, \frac{2x^2 - 2}{x - 1} = 2(x + 1)$. The definition of a limit requires that

$$\left| \frac{2x^2 - 2}{x - 1} - 4 \right| < \varepsilon \quad \left(\textit{whenever} \quad 0 < \left| x - 1 \right| < \delta \right)$$

$$\Rightarrow \quad \left| 2(x + 1) - 4 \right| < \varepsilon \quad \Rightarrow \quad 2\left| (x + 1) - 2 \right| < \varepsilon \quad \Rightarrow \quad \left| x - 1 \right| < \frac{\varepsilon}{2}$$

So by setting $\delta = \varepsilon/2$, we construct the required δ for any given ε. Hence, for every ε, there exists a δ satisfying the definition of a limit.

B. Theorems which make the definition easy to apply (a a constant; f, g, h functions):

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \left(\lim_{x \to a} f(x) \right) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \left[f(x) g(x) \right] = \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} g(x) \right)$$

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{if this fraction is defined}$$

L'Hôpital's rule: If $\frac{f(a)}{g(a)}$ is indeterminate $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

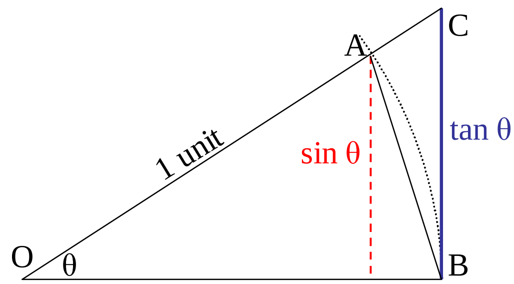
Example: $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \to \left(\frac{0}{0} \right)$, *so* $\quad \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{1} = 0$

The Squeeze Theorem: If

$$f(x) \leq g(x) \leq h(x), \textit{ and } \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \textit{ then } \lim_{x \to a} g(x) = L.$$

Example: Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

The diagram shows a section of the unit circle. Comparing the areas of triangle OAB with circular segment OAB, we see



$$\sin\theta < \theta \qquad \Rightarrow \qquad \frac{\sin\theta}{\theta} < 1 .$$

Comparing segment OAB with triangle OCB:

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \Rightarrow \qquad \cos\theta < \frac{\sin\theta}{\theta} .$$

Given that $\lim_{\theta \to 0} \cos\theta = 1$, and noting that the inequalities apply for both small positive and negative θ, we apply the squeeze theorem:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1, \ \textit{and} \ \lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} 1 = 1 \quad \Rightarrow \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

C. Infinite Limits: Definitions:

$$L = \lim_{x \to \infty} f(x) \text{ iff } \forall \varepsilon > 0, \exists M \ni \left| f(x) - L \right| < \varepsilon \text{ whenever } x > M$$

$$\lim_{x \to a} f(x) \to \infty \text{ iff } \forall N, \exists \delta \ni f(x) > N \text{ whenever } 0 < \left| x - a \right| < \delta$$

$$\lim_{x \to \infty} f(x) \to \infty \text{ iff } \forall N, \exists M \ni f(x) > N \text{ whenever } x > M$$

Example: $\lim_{\theta \to \infty} \sin\theta$ does not exist (is not finite), and is not infinite.

Derivatives and integrals are discussed below for each case separately.

III. f: $R^1 \rightarrow R^1$

Differential Calculus

A. Definition: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Examples:

$$\frac{d(x^2)}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} 2x + h = 2x$$

$$\frac{d(\sin\theta)}{d\theta} = \lim_{h\to 0} \frac{\sin(\theta+h) - \sin(\theta)}{h} = \lim_{h\to 0} \frac{\sin\theta\cos h + \cos\theta \sin h - \sin(\theta)}{h}$$

$$= \lim_{h\to 0} \cos\theta \frac{\sin h}{h} = \cos\theta$$

All trigonometric derivative formulas follow from that for sin θ.

B. Theorems which make the definition easy to apply (a, b constants; $f(x)$, $g(x)$ functions):

$(af + bg)' = af' + bg'$

$(fg)' = f \cdot g' + f' \cdot g$ (product rule)

$\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$ (quotient rule)

$[f(g(x))]' = f'(g(x)) \cdot g'(x)$ (chain rule)

Example: Chain rule:	$f(x) = \sin x$	$f'(x) = \cos x$
	$g(x) = x^2$	$g'(x) = 2x$
	$f(g(x)) = \sin x^2$	$[f(g(x))]' = f'(g(x)) \cdot g'(x) = (\cos x^2)(2x)$

C. The derivative approximates the change in the value of a function as a linear function of the change in its argument (the **differential**).

$\Delta f \approx f'(a)\,\Delta x$ E.g., $f(x) = x^2$ => $\Delta f \approx 2x\,\Delta x$

near $x = 3$: $f(3) = 9$, $f'(3) = 6$, $f(x) - 9 \approx 6(x - 3)$

D. Taylor's Theorem: using higher derivatives, one can construct better (quadratic, cubic, etc.) approximations to a function at a point. Expanded about a:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(a,x)$$

$$R_n(a,x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \qquad a \le c \le x$$

Example: Expanding e^x about $x = 0$:

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + R_n(0,x)$$

$$\left|R_n(0,x)\right| = \frac{e^c}{(n+1)!} x^{n+1} \le e^c \left|\frac{x^{n+1}}{(n+1)!}\right|$$

Integral Calculus

A. Definition:

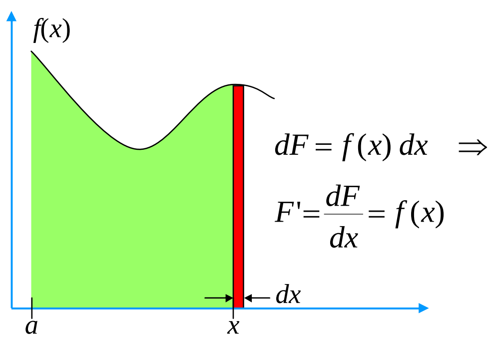
$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{N} f(\xi_i)\,\Delta_i x$$

$$\Delta = \{x_0 = a, x_1, x_2, x_3, \ldots x_N = b\}, \qquad \Delta_i x = x_{i+1} - x_i\,, \qquad x_i \le \xi_i \le x_{i+1}$$

B. Theorems which make the definition easy to apply:

$$F(x) = \int_a^x f(x)\,dx \quad \Rightarrow \quad F'(x) = f(x)$$

(Fundamental Theorem of Calculus)



$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

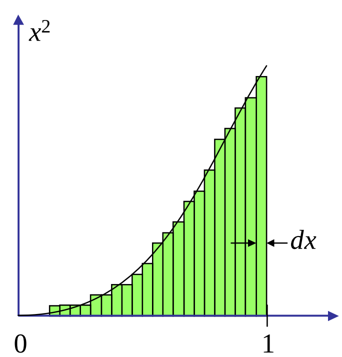
(change of variable)

Example: Change of variable:

$$\int_0^\pi (1+\cos\theta)^{3/4} \sin\theta\,d\theta = \int_0^\pi -(1+\cos\theta)^{3/4} d(1+\cos\theta) = -\frac{4}{7}(1+\cos\theta)^{7/4}\Big|_0^\pi = \frac{4}{7}2^{7/4} = \frac{2^{15/4}}{7}$$

C. Integral = limit of a sum of pieces which approximate the quantity of interest. The limit is taken as the pieces get smaller and more numerous. To get a useful result, the approximation must be perfect for infinitely many infinitesimal pieces.

Example: $\displaystyle\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$



D. Advanced techniques of evaluation:

1. Trigonometric substitution:

$$\int \sqrt{1-x^2}\ dx, \quad let\ x = \sin\theta, \quad \Rightarrow dx = \cos\theta\ d\theta$$

$$\int \frac{1}{x^2+2x+6}\ dx$$

2. Partial fractions: $\int \frac{1}{x^2-1}\ dx$

3. Integration by parts (product rule in reverse): $\int U\ dV = UV - \int V\ dU$

Example:	$\int_0^1 x\,e^x\ dx \qquad \left(\text{Let } U = x \Rightarrow dU = dx, \qquad dV = e^x dx \Rightarrow V = e^x\right)$ $\int_0^1 x\,e^x\ dx = \left[xe^x\right]_0^1 - \int_0^1 e^x\ dx = \left[xe^x - e^x\right]_0^1 = 1$

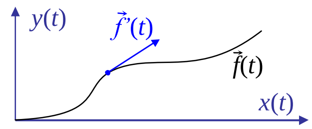
E. **Improper Integrals**: Definition: If f is not continuous at a,

$$\int_a^b f(x)\ dx \equiv \lim_{x\to a} \int_x^b f(x)\ dx$$ (and similarly if f discontinuous at b, or both)

IV. f: $R^1 \rightarrow R^2$

$\vec{f}(t) = (x(t), y(t))$, in other words, *f* is a collection (vector) of two functions, *x*(*t*) and *y*(*t*).

Differential Calculus



A. Definition: $\vec{f}'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$

B. $\vec{f}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (\dot{x}, \dot{y})$ = velocity vector = tangent to curve

$$\left\| \vec{f}' \right\| = \sqrt{\dot{x}^2 + \dot{y}^2} = speed$$

$$\text{unit tangent} = \frac{velocity}{speed} = \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)$$

$$f''(t) = (\ddot{x}, \ddot{y}) = \text{acceleration vector}$$

Chain Rule (change of parameter):

$$\vec{f}(t(s))' = \vec{f}'(t(s)) \cdot t'(s) = \frac{d\vec{f}}{dt} \cdot \frac{dt}{ds} = \left(\frac{dx}{dt}\frac{dt}{ds}, \frac{dy}{dt}\frac{dt}{ds} \right)$$

C. The derivative approximates the change in the value of a function as a linear function of the change in its argument (the **differential**).

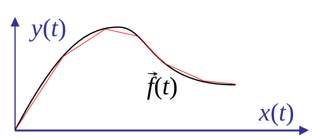
$$\Delta\vec{f} = (\Delta x, \Delta y) \approx \vec{f}'(a)\Delta t$$

D. Taylor's Theorem: using higher derivatives, one can construct better (quadratic, cubic, etc.) approximations to a function at a point. Expanded about *a*:

$$\vec{f}(t) = \vec{f}(t) + \frac{\vec{f}'(a)}{1!}(t-a) + \frac{\vec{f}''(a)}{2!}(t-a)^2 + \ldots$$

$$(x(t), y(t)) = (x(a), y(a)) + \frac{(\dot{x}(a), \dot{y}(a))}{1!}(t-a) + \frac{(\ddot{x}(a), \ddot{y}(a))}{2!}(t-a)^2 + \ldots$$

Integral Calculus



A. s = arc length = $\int \left| d\vec{f} \right| = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{N} \left| \vec{f}(t_{i+1}) - \vec{f}(t_i) \right|$

B.

$$s = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \text{time integral of speed}$$

$$= \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

C. **Exercises**:

1. $(x, y, z) = (t, t^2, 1)$

   velocity($t = 1$) =

   speed =

   distance traveled t=0 to t=1 ?

   unit tangent at t=1 ?

2. Find length of $y = x^2$ between $x = 0$ and $x = 1$.

V. f: $R^2 \rightarrow R^1$

$z = f(x, y)$

Differential Calculus

A. Definition:

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \qquad \frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$f' = Df = \nabla f = gradient = grad\ f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$f'' = D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\, \partial x} \\ \frac{\partial^2 f}{\partial x\, \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

∇f is perpendicular to level curves (curves such that $f(x, y)$ = constant)

∇f points in the direction of fastest (steepest) increase of *f*.

B. $D_{\vec{a}} f$ = directional derivative of *f*

= rate of change of *f* per unit distance in the direction $\vec{a}$.

$$= \nabla f \cdot \vec{a} \qquad where\ \|\vec{a}\| = 1,\ i.e.,\ \vec{a} = \frac{\vec{v}}{\|\vec{v}\|}$$

Maximum value of $D_{\vec{a}} f = \|\nabla f\| = \sqrt{f_x{}^2 + f_y{}^2}$

Chain rule:

$$z = f\left(x(t), y(t)\right) = f\left(\vec{g}(t)\right)$$

$$\frac{dz}{dt} = \left(\nabla f(x, y)\right)(\dot{x}, \dot{y}) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= f'\left(\vec{g}(t)\right) \cdot \vec{g}'(t)$$

C. The derivative approximates the change in the value of a function as a linear function of the change in its argument (the **differential**).

$$\Delta f \approx \nabla f \cdot \left(\Delta x, \Delta y\right) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

D. Tangent plane:

$$\vec{N}\cdot\left(X - X_0\right)=0$$
$$\left(f_x, f_y, -1\right)\cdot\left(x-a, y-b, z-f(a,b)\right)=0$$

Normal line:

$$X = X_0 + t\vec{N}$$

$$\left(x,y,z\right)=\left(a,b,f(a,b)\right)+t\left(f_x, f_y,-1\right)$$

E. Taylor's Theorem, expanded about (a,b):

$$f(x,y) = f(a,b)+\frac{\nabla f\left(a,b\right)}{1!}\cdot(x-a, y-b)+\frac{[x-a,y-b]\left[D^2 f\right]}{2!}\cdot(x-a,y-b)+\ldots$$
$$= f(a,b)+\frac{\partial f}{\partial x}(x-a)+\frac{\partial f}{\partial y}(y-b)+\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x-a)^2+2\frac{\partial^2 f}{\partial x\,\partial y}(x-a)(y-b)+\frac{\partial f}{\partial y}(y-b)^2\right)+\ldots$$

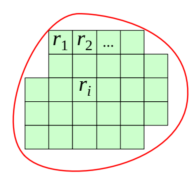
F. **Exercises**:

1. Find $D_{(1,0,0)}f(x, y, z)$
2. $\nabla(xy - z^2 \cos x) =$
3. Tangent plane to $x^2 + y^2 = z$, at (1, 1, 2)
4. Maximum value of $f(x, y) = 9 - x^2 + 2x - y^2 + 6y$
5. Minimum of $x^2 + y^2$, given $x^2 - y = 1$

Integral Calculus

A. Definition, integral over a region F:

$$\iint_F f(x,y)\,dA = \lim_{\|\Delta\|\to 0}\sum_i f(\xi_i,\eta_i)A(r_i), \qquad (\xi_i,\eta_i)\in(r_i)$$



B. $$\iint_F f\,dA = \int\left(\int f\,dx\right)dy = \int\left(\int f\,dy\right)dx$$

Exercise:

$$F = \{(x,y);\ 1\le x\le 2,\ \ x\le y\le 2\}, \qquad f(x,y)=x\ln y$$

$$\int\int_F x\ln y\, dA =$$ (do both ways)

C. Change of variable (the "anyway you slice it" theorem):

$$g(x,y) = (u,v) \qquad (x,y) = g^{-1}(u,v)$$

$$\iint_F f(u,v)\,du\,dv = \iint_{g^{-1}(F)} f\big(g(x,y)\big)\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

$$= \iint_{g^{-1}(F)} f\big(g(x,y)\big)\left|Dg\right| dx\,dy = \iint_{g^{-1}(F)} f\big(g(x,y)\big)\frac{1}{\left|Dg^{-1}\right|} dx\,dy$$

Exercises:

1.

$$\vec{g}(x,y) = (r,\theta) = \left(\sqrt{x^2+y^2},\ \arctan\frac{y}{x}\right)$$

$$(x,y) = g^{-1}(r,\theta) = (r\cos\theta, r\sin\theta) \qquad dx\,dy = r\,dr\,d\theta\,??$$

2.

$$\vec{h}(x,y,z) = (r,\theta,z) =$$

$$(x,y,z) = h^{-1}(r,\theta,z) = \qquad dx\,dy\,dz =$$

3.

$$\vec{j}(x,y,z) = (\rho,\theta,\phi) =$$

$$(x,y,z) = j^{-1}(r,\theta,z) = \qquad dx\,dy\,dz =$$

4. Find $\iint_F \sqrt{x^2+y^2}\,dx\,dy, \quad where\ F = \{(r,\theta);\ r \le \theta,\ 0 \le \theta \le \pi\}$

5. Find the center of mass of the cone

$$\left\{(x,y,z);\ \sqrt{x^2+y^2} \le z \le 1\right\}, \qquad \delta = x^2 + y^2 - z^2 + 1$$

6. Find the volume of the paraboloid $z = x^2 + 2y^2$, below $z = 1$.

7. Find the volume of $R = \{(x,y,z);\ 0 \le x \le 1,\ 1 \le y \le z,\ 1 \le z \le 2\}$

VI. f: $R^n \to R^m$

$f(x, y) = (u, v) = (u(x, y), v(x, y))$

Differential Calculus

A. Definition:

$$\vec{f}' = Df = \text{Derivative of } \vec{\text{f}} = \text{Jacobian of } \vec{\text{f}} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

B. Chain rule, Case 1: $f : \mathrm{R}^3 \to \mathrm{R}^3,\ g : \mathrm{R}^3 \to \mathrm{R}^3$ $\quad \left[\vec{f}(\vec{g}(\vec{x}))\right]' = \vec{f}'(\vec{g}(\vec{x})) \cdot \vec{g}'(\vec{x})$

Example: $\frac{\partial(\rho,\theta,\phi)}{\partial(x,y,z)} = \frac{\partial(\rho,\theta,\phi)}{\partial(r,\theta,z)} \cdot \frac{\partial(r,\theta,z)}{\partial(x,y,z)}$

Chain rule, Case 2: $f : \mathrm{R}^3 \to \mathrm{R}^1,\ g : \mathrm{R}^3 \to \mathrm{R}^3$

$\vec{g}(r,s,t) = (x,y,z)$

$[f(\vec{g}(r,s,t))]' = f'(\vec{g}(r,s,t)) \cdot \vec{g}'(r,s,t)$

$$= \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix}$$

$$= \left(f_x x_r + f_y y_r + f_z z_r,\ f_x x_s + f_y y_s + f_z z_s,\ f_x x_t + f_y y_t + f_z z_t\right)$$

Exercise: $f(x,y) = xe^y$, $\quad (x,y) = (s^2 + t, \cos t)$, Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$

C. $\Delta\vec{f} = (\Delta u, \Delta v) \approx f'(a,b)(\Delta x, \Delta y)$

D. Taylor's Theorem, expanded about (a,b):

$$\vec{f}(x,y) = f(a,b) + \frac{D\vec{f}(a,b)}{1!} \cdot (x-a, y-b) + \dots$$

VII. Convergence of Infinite Series

Arbitrary: $\sum_{n=0}^{\infty} u_n$; or power series: $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} a_n x^n$

A. Ratio test: Arbitrary: $\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| \Rightarrow$ Power series: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \cdot |x|$

If $\rho < 1$: Converges absolutely

$\rho > 1$: Diverges absolutely

$\rho = 1$: No information

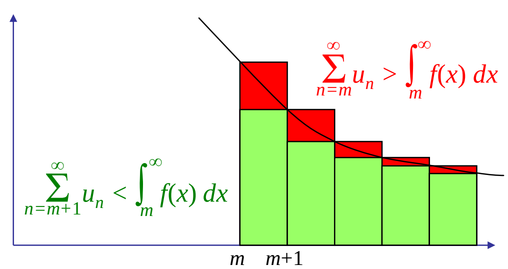
Which implies a power series converges for $|x| < \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

B. Comparison tests:

(1) Compare u_n to a known convergent series, v_n. If there exist m and n such that $u_{m+i} \le v_{n+i}$ for all $i \ge 0$, then Σu_n converges.

(2) Compare u_n to a known divergent series, v_n. If there exist m and n such that $u_{m+i} \ge v_{n+i}$ for all $i \ge 0$, then Σu_n diverges.

C. Integral test: If u_n can be written as a decreasing function on the reals, $f(x)$, such that $f(n) = u_n$, for all $n \ge m$, then the series Σu_n has the same convergence or divergence as the integral $\int_m^\infty f(x)\,dx$.



D. Alternating series test: For a series whose terms alternate in sign, the series converges iff $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$.

How to Edit/Update This Document

This document was created with Microsoft Word XP under Windows 2000, and its included Equation Editor 3.0. To edit the document, you should view it in "Normal View." Open the Tools / Options / View dialog box, and set "Style area width: to 0.8" or so. Set "Field shading" to "Always." This will display each paragraph style to the left of the paragraph, and all equation objects will be shaded on-screen (but not in the printed document). To make changes or additions, copy entire paragraphs (including their styles) that are similar to the new ones, and then edit those copies.

In particular, note that the table of contents is automatically generated. Therefore, you must use the proper paragraph style for chapter titles. To add a new chapter, copy the title of an existing chapter, and then edit the text.

This document uses exactly 3 fonts: Times New Roman, Symbol, and Arial. Times New Roman is the workhorse of all paragraphs, and should be available on most any computer, even Apple. It is the source of all Greek letters (see "bugs" below). Symbol has been a part of Windows since the beginning, and is necessary for its mathematical symbols. Arial is purely for decorating the chapter titles, and can be substituted with anything you like.

There are 2 spaces between sentences. Please be consistent.

The paragraphs in each chapter are deliberately numbered by hand. This is because the automatic numbering schemes in Word get very confused by anything beyond trivial paragraphs. It's actually easier to maintain the numbers by hand than to contort Word to do it "automatically."

Simple, one-line equations can be entered directly in Word, including Greek letters and sub- or super-scripts. Complex equations, with summations, matrices, simultaneous sub- & super- scripts must use the Equation Editor. To force a space in the Equation Editor, use Ctrl-Space (narrow space), or Ctrl-Shift-Space (wide space). In particular, despite on-screen appearances, the limits of a definite integral are smashed (by default) into the integral sign. Precede each integral limit with a wide space to make it look normal. Also, see the matrices for examples of difficult formatting, and equation spacing tricks.

MS-Word breaks text (wraps lines) on any space or hyphen. Sometimes, this is undesirable: you don't want the formula "*a-b*" to end up with "*a-*" at the end of a line, and "*b*" at the beginning of the next. To achieve this, use a "non-breaking" hyphen: Ctrl-Shift-hyphen. It looks like a hyphen, but won't allow a line break on it. Similarly, you can enter a non-breaking space with Ctrl-Shift-space, because you wouldn't really enter "*a-b*"; you'd space it out to look better: "*a* - *b*".

A hyphen is too short for a negative sign (- *A*); use Ctrl-Numeric-hyphen for a longer one: (– *A*).

Microsoft bugs: Despite the promise of "TrueType," what you see is not always what you get. In particular, the Times New Roman glyph for the Greek letter "phi" appears on-screen as an old-style phi, but prints on my HP Laserjet 4100 as a modern phi. Most math texts treat the two styles of phi as if they were different letters, and many use them simultaneously to mean different things. This is not possible when you can't tell what will print from what you see.

Though Microsoft claims that Word documents can be "seamlessly" transported between platforms and operating systems, that has never been true. Especially with a document containing obscure features, like math symbols and Equations, there is virtually no chance you can successfully convert this document to any other platform or text format.

Word crashes frequently with equations, and as a result, some of the equations are now "pictures" which cannot be edited with Equation Editor. Real editable equations are shaded on-screen as a field (if you followed the instructions above). The "pictures" are not shaded as a field. If you need to change such a corrupted equation, you must enter it anew in Equation Editor. Again, copy some similar equation, and then edit it. In particular, if you select the whole document, update fields, and save the document, Word always crashes.

Contact the Justice Department for a resolution to all these problems, since only monopolies can survive with such consistently poor quality products.

This document provides an "Italicize" macro, and a "Math" button which invokes it. This macro makes all alphabetic characters in the selection italic, without affecting other characters, such as numbers. This is standard formatting for mathematics text. So you can just type the formulas without worrying about the italics, then select the whole formula and click "math".

