# Problem of the Fortnight

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Using Heron's Formula with a bit of algebra (i.e. factoring a polynomial over the rationals in Magma) we can see that the area of such a triangle is $\sqrt{\frac{3}{16}(n-1)(n+1)^2(n+3)}$. Setting $m = n+1$, we search for integers $m$ such that $\frac{3}{16}(m-2)m^2(m+2) = \frac{3}{16}m^2(m^2-4)$ is a square integer. Notice that if $m$ is odd, there is no hope that it can even be an integer, hence we set $m = 2u$. And we now wish to find natural numbers $u$ such that $3(u^2-1)$ is square. Suppose $3(u^2-1) = s^2$. Since 3 is prime, it must divide s. Hence we have that $u^2 - 1 = 3t^2$ for some integer $t$. Or equivalently, $u^2 - 3t^2 = 1$, which is a particular case of Pell's Equation [1]. Using an online calculator [2] we obtain $u = 2$ and $t = 1$ to be the fundamental solution. Observe that once we have obtained suitable $u$ and $t$, we can recover the area of the triangle by noting that $\frac{3}{16}(m+2)m^2(m+2) = 3u^2(u-1)^2 = 3u^2(3t^2)$. Hence the area of the triangle is $3ut$. We can now write a little Magma code to generate the solutions, check if the area is divisible by 20 and finally confirm if one the sides is indeed a prime number

```
u := 2; t := 1;
for i := 1 to 15 do;
M := 3 * u * t;
if IsDivisibleBy( M , 20 ) then
u; t; M; 2*u - 1; IsPrime(2*u - 1); IsPrime(2*u + 1);
end if;
a := u; b := t;
//observe this recurrence in "Solution Technique" of [1]
u := 2*a + 3*b;
t := 2*b + a;
end for;
```

The magma code returns the following output

```
1351
780
3161340
2701
false
false
3650401
2107560
23080317394680
7300801
true
false
```

Hence we have that the smallest $n$ for which the triangle with sides $n, n+1, n+2$ satisfies the given conditions is 7300801. The area of the triangle is 23080317394680. I used a for loop with 15 iterations simply to avoid running some monstrous while loop. Luckily the solution pops out very quickly. I'd still really like to find a more analytical solutin to this, but this will have to do for now.

**1** http://en.wikipedia.org/wiki/Pell%27s_equation
**2** http://www.bioinfo.rpi.edu/zukerm/cgi-bin/dq.html