Chapter 6
Sensitivity Analysis and Duality

to accompany
Introduction to Mathematical Programming: Operations Research, Volume 1
4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

Presentation: H. Sarper
6.5 – Finding the Dual of an LP

Associated with any LP is another LP called the dual. Knowledge of the dual provides interesting economic and sensitivity analysis insights.

When taking the dual of any LP, the given LP is referred to as the primal. If the primal is a max problem, the dual will be a min problem and visa versa.

Define the variables for a max problem to be $z, x_1, x_2, \ldots, x_n$ and the variables for a min problem to be $w, y_1, y_2, \ldots, y_n$.

Finding the dual to a max problem in which all the variables are required to be nonnegative and all the constraints are $\leq$ constraints (called normal max problem) is shown on the next slide.
6.5 – Finding the Dual of an LP

Normal max problem

It’s dual

Normal min problem

It’s dual

max \( z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \)

s.t.

\( a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \)

\( a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \)

\( \ldots \ldots \ldots \ldots \ldots \)

\( a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \)

\( x_j \geq 0 \) (j = 1, 2, \ldots, n)

min \( w = b_1y_1 + b_2y_2 + \ldots + b_my_m \)

s.t.

\( a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1 \)

\( a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2 \)

\( \ldots \ldots \ldots \ldots \ldots \)

\( a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n \)

\( y_i \geq 0 \) (i = 1, 2, \ldots, m)
6.5 – Economic Interpretation of the Dual Problem

Interpreting the Dual of the Dakota (Max) Problem

The primal is: \[ \text{max } z = 60x_1 + 30x_2 + 20x_3 \]
\[
\text{s.t. } 8x_1 + 6x_2 + x_3 \leq 48 \quad \text{(Lumber constraint)} \\
4x_1 + 2x_2 + 1.5x_3 \leq 20 \quad \text{(Finishing constraint)} \\
2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \quad \text{(Carpentry constraint)} \\
x_1, x_2, x_3 \geq 0
\]

The dual is: \[ \text{min } w = 48y_1 + 20y_2 + 8y_3 \]
\[
\text{s.t. } 8y_1 + 4y_2 + 2y_3 \geq 60 \quad \text{(Desk constraint)} \\
6y_1 + 2y_2 + 1.5y_3 \geq 30 \quad \text{(Table constraint)} \\
y_1 + 1.5y_2 + 0.5y_3 \geq 20 \quad \text{(Chair constraint)} \\
y_1, y_2, y_3 \geq 0
\]
6.5 – Economic Interpretation of the Dual Problem

The dual is: \[ \text{min } w = 48y_1 + 20y_2 + 8y_3 \]
\[ \text{s.t. } 8y_1 + 4y_2 + 2y_3 \geq 60 \quad \text{(Desk constraint)} \]
\[ 6y_1 + 2y_2 + 1.5y_3 \geq 30 \quad \text{(Table constraint)} \]
\[ y_1 + 1.5y_2 + 0.5y_3 \geq 20 \quad \text{(Chair constraint)} \]
\[ y_1, y_2, y_3 \geq 0 \]

Relevant information about the Dakota problem dual is shown below.

<table>
<thead>
<tr>
<th>Resource</th>
<th>Desk</th>
<th>Table</th>
<th>Chair</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber</td>
<td>8 board ft</td>
<td>6 board ft</td>
<td>1 board ft</td>
<td>48 boards ft</td>
</tr>
<tr>
<td>Finishing</td>
<td>4 hours</td>
<td>2 hours</td>
<td>1.5 hours</td>
<td>20 hours</td>
</tr>
<tr>
<td>Carpentry</td>
<td>2 hours</td>
<td>1.5 hours</td>
<td>0.5 hours</td>
<td>8 hours</td>
</tr>
<tr>
<td>Selling Price</td>
<td>$60</td>
<td>$30</td>
<td>$20</td>
<td></td>
</tr>
</tbody>
</table>
6.5 – Economic Interpretation of the Dual Problem

The first dual constraint is associated with desks, the second with tables, and the third with chairs. Decision variable $y_1$ is associated with lumber, $y_2$ with finishing hours, and $y_3$ with carpentry hours.

Suppose an entrepreneur wants to purchase all of Dakota’s resources. The entrepreneur must determine the price he or she is willing to pay for a unit of each of Dakota’s resources.

To determine these prices we define:

- $y_1 =$ price paid for 1 boards ft of lumber
- $y_2 =$ price paid for 1 finishing hour
- $y_3 =$ price paid for 1 carpentry hour

The resource prices $y_1$, $y_2$, and $y_3$ should be determined by solving the Dakota dual.
The total price that should be paid for these resources is $48y_1 + 20y_2 + 8y_3$. Since the cost of purchasing the resources is to minimized:

$$\min w = 48y_1 + 20y_2 + 8y_3$$

is the objective function for the Dakota dual.

In setting resource prices, the prices must be high enough to induce Dakota to sell. For example, the entrepreneur must offer Dakota at least $60 for a combination of resources that includes 8 board feet of lumber, 4 finishing hours, and 2 carpentry hours because Dakota could, if it wished, use the resources to produce a desk that could be sold for $60. Since the entrepreneur is offering $8y_1 + 4y_2 + 2y_3$ for the resources used to produce a desk, he or she must chose $y_1$, $y_2$, and $y_3$ to satisfy: $8y_1 + 4y_2 + 2y_3 \geq 60$
Similar reasoning shows that at least $30 must be paid for the resources used to produce a table. Thus $y_1$, $y_2$, and $y_3$ must satisfy:

$$6y_1 + 2y_2 + 1.5y_3 \geq 30$$

Likewise, at least $20 must be paid for the combination of resources used to produce one chair. Thus $y_1$, $y_2$, and $y_3$ must satisfy:

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$

The solution to the Dakota dual yields prices for lumber, finishing hours, and carpentry hours.

In summary, when the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as resource shadow prices.