Math 534A Problem Set on Quotient Sets

1. Given the set \( S = \{a, b, c, d, e\} \) and an equivalence relation \( \cong \) such that \( a \cong e \), \( b \cong d \), and \( d \cong c \), find the quotient set \( S/\cong \).

2. Let \( \mathcal{N} \) be the set of natural numbers, which for the purposes of this problem start at 0. Define the relation \( \cong \) on pairs of natural numbers by \((a, b) \cong (c, d)\) iff \( a + d = b + c \).

   a. Show that this defines an equivalence relation on \( \mathcal{N} \times \mathcal{N} \).

   b. Show that for every equivalence class mod \( \cong \), there is exactly one \( a \in \mathcal{N} \) such that the equivalence class has an element of the form \((a, 0)\) or \((0, a)\).

   c. Define \( \mathcal{Z} = \mathcal{N} \times \mathcal{N}/\cong \). Identify non-negative integers with equivalence classes with an element of the form \((a, 0)\) and non-positive integers with those that have an element of the form \((0, a)\).

      Show that addition defined \( \mathcal{N} \times \mathcal{N} \) by \((a, b) + (c, d) = (a + c, b + d)\), is well defined on \( \mathcal{Z} \) by showing that if \((a, b) \cong (x, y)\) and \((c, d) \cong (u, v)\) then \((a + c, b + d) \cong (x + u, y + v)\), i.e. show that if we use any elements of the equivalence classes then we get equivalent results.