Math 534A Solution to exercise 3.1.1

Show that $A \subset M$ is sequentially compact iff every infinite subset of A has an accumulation point.

$\Longrightarrow$

Assume A sequentially compact and consider $S \subset A$, $|S| = \infty$. If no such S exists, the result is vacuously true. Construct an injective sequence x_n by choosing successive elements so that $x_{n+1} \in S - \{x_1, x_2, ...x_n\}$. This is possible since S is infinite. Since A is sequentially compact and the sequence x_n lies in A, the sequence x_n has a convergent subsequence $x_{k_n} \to x \in A$. Then x is an accumulation point of S since convergence implies that for all $\epsilon > 0$, there exists N such that $n > N \Rightarrow x_{k_n} \in D(x, \epsilon)$ and at most one of these x_{k_n} can equal x.

$\Longleftarrow$

Now assume all infinite subsets of A have an accumulation point and consider any sequence x_n in A. Let $S = \text{Range}(x_n)$. If S is finite, x_n takes on some value infinitely often, and thus has a convergent subsequence (Proof?). If S is infinite, it has an accumulation point x by our hypothesis. Then every $D(x, 1/n)$ has infinitely many points of S so we can define a sequence k_n (of increasing integers!) such that x_{k_n} is in $D(x, 1/n) - \{x_1, x_2, ..., x_{k_{n-1}}\}$. This subsequence converges to x.