1) Let $A$ be a $6 \times 6$ matrix with characteristic equation

$$\lambda^2(\lambda+1)(\lambda-7)^3 = 0.$$ What are the possible dimensions for eigenspaces of $A$?

By definition, if $\lambda_0$ is an eigenvalue of an $n \times n$ matrix $A$, then the dimension of the eigenspace corresponding to $\lambda_0$, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of $A$ is called the algebraic multiplicity of $\lambda_0$.

We know that, if $A$ is a square matrix, then for every eigenvalue of $A$, the geometric multiplicity is less than or equal to the algebraic multiplicity.

Let us denote the dimension of the eigenspace as $D$. We also know each eigenvalue corresponds to at least one eigenvector. Thus, in our case we have:

- For $\lambda = 0$: $D \leq 2$
- For $\lambda = 1$: $D = 1$
- For $\lambda = 7$: $D \leq 3$

2) Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of $A$.

The characteristic polynomial of $A$ is:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & 0 & -1 \\ -2 & \lambda - 3 & -2 \\ -1 & 0 & \lambda - 4 \end{bmatrix}$$

$$= (\lambda - 3)(\lambda - 5)$$

So, for eigenvalues, $\det(\lambda I - A) = 0$. 

$$\det(\lambda I - A) = 0$$
(b) For each eigenvalue $\lambda$, find the rank of the matrix $\lambda I - A$.

For $\lambda = 3$

\[
(3I - A) = \begin{bmatrix}
3 & 0 & -1 \\
-2 & 0 & -2 \\
1 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Similarly, for $\lambda = 5$.

For $\lambda = 3$:

\[
\lambda I - A = \begin{bmatrix}
-1 & 0 & -1 \\
-2 & 0 & -2 \\
1 & 0 & -1 \\
\end{bmatrix} : \text{rank of matrix: 1}
\]

For $\lambda = 5$:

\[
\lambda I - A = \begin{bmatrix}
1 & 0 & -1 \\
-2 & 0 & -2 \\
1 & 0 & 1 \\
\end{bmatrix} : \text{rank of matrix: 2}
\]

For $\lambda = 3, 5$: matrix is not diagonalizable.

(c) Is $A$ diagonalizable? Justify your conclusion.

Yes, because the matrix $A$ has $m$ distinct eigenvalues.

Therefore, $A$ is diagonalizable.
8) Find a matrix \( P \) that diagonalizes \( A \), and determine \( P^{-1}AP \).

\[
A = \begin{bmatrix}
-14 & 12 \\
-20 & 17
\end{bmatrix}
\]

\[
\text{det}(P - A) = \begin{vmatrix}
\lambda + 14 & -12 \\
620 & \lambda - 17
\end{vmatrix}
\]

\[
= (\lambda + 14)(\lambda - 17) + 240
\]

\[
= \lambda^2 - 3\lambda + 2
\]

Solving \( \lambda^2 - 3\lambda + 2 = 0 \)

\[
\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0
\]

\[
\Rightarrow (\lambda - 2)(\lambda - 1) = 0
\]

Yields the following eigenvalues and corresponding eigenvectors:

\( \lambda = 1 \):
\[ p_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \]

\( \lambda = 2 \):
\[ p_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \]

There are two basis vectors in total, so matrix \( A \) is diagonalizable and

\[
P = \begin{bmatrix}
4 & 3 \\
5 & 4
\end{bmatrix}
\]

diagonalizes \( A \).

We know that in this case the matrix \( P^{-1}AP \) will be diagonal with \( \lambda_1, \lambda_2, \ldots, \lambda_n \) as its successive diagonal entries, where \( \lambda_n \) is the eigenvalue corresponding to \( p_i \), for \( i = 1, 2, \ldots, n \). Thus

\[
P^{-1}AP = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

\[
P^{-1}AP = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

10) Find a matrix \( P \) that diagonalizes \( A \), and determine \( P^{-1}AP \).
Let us find the eigenvalues of the given matrix first. The characteristic polynomial of $A$ is
\[
\det (M - A) = \det \begin{bmatrix}
\lambda - 1 & 0 & 0 \\
0 & \lambda - 1 & 1 \\
0 & 1 & \lambda - 1
\end{bmatrix} = (\lambda - 1)(\lambda^2 - 2\lambda) = 0
\]
Solving this equation $\lambda^2 - 2\lambda = 0$ yields the following eigenvalues and corresponding eigenvectors:

$\lambda = 0$: $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \lambda = 1$: $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 2$: $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

There are three basis vectors in total, so matrix $A$ is diagonalizable and
\[
P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}
\]
diagonalizes $A$.

We know that in this case the matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where $\lambda_i$ is the eigenvalue corresponding to $v_i$, $i = 1, 2, \ldots, n$. Thus, we obtain
\[
P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\]
15) Find the geometric and algebraic multiplicity of each eigen value, and determine whether A is diagonalizable. If so, find a matrix P that diagonalizes A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

The given matrix is a lower triangular matrix, thus by inspection we find the eigen values: 

- $\lambda = 0$: Algebraic multiplicity equals 2.
- $\lambda = 1$: Algebraic multiplicity equals 1.

Let us find the eigenvectors and thus determine the geometric multiplicity of the eigenvalues. We have

$$\lambda = 0: \quad P_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 1: \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus we see that for every eigenvalue the geometric multiplicity is equal to the algebraic multiplicity. So, A is diagonalizable, and

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

diagonalizes A.

We know that in this case, matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where $\lambda_n$ is the eigenvalue corresponding to $P_i$ for $i = 1, 2, \ldots, n$. Thus
Thus, we obtain
\[ p^T A p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ p = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad p^T A p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

17) Find the geometric and algebraic multiplicities of each
eigenvalue, and determine whether \( A \) is diagonalizable.
If so, find a matrix \( p \) that diagonalizes \( A \), and
determine \( p^T A p \).

\[ A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \]

The given matrix is an upper triangular matrix. Thus, by inspection we find the eigenvalues:
\[ \lambda = -2: \text{algebraic multiplicity equals 2}. \]
\[ \lambda = 3: \text{algebraic multiplicity equals 2}. \]

Let us find the eigen vectors and determine the
geometric multiplicities of the eigen values. We have:
\[ \lambda = -2: p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \lambda = 3: p_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad p_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

Thus we see that for every eigenvalue the geometric
multiplicity is equal to the algebraic multiplicity. So \( A \) is
diagonalizable and
\[ p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
diagonalizes $A$.

We know that in this case the matrix $P^T A P$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where $\lambda_i$ is the eigenvalue corresponding to $p_i$ for $i = 1, 2, \ldots, n$.

Thus we obtain

$$P^T A P = \begin{bmatrix}
-2 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}$$

18) Use the method of example 6 to compute $A^{10}$, where

$$A = \begin{bmatrix}
1 & 0 \\
-1 & 2
\end{bmatrix}$$

We know that

$$A^{10} = P D^{10} P^{-1}$$

where $D$ is a diagonal matrix.

Let us find the diagonal matrix.

The given matrix is lower diagonal matrix. Thus by inspection we find the eigenvalues and the corresponding eigenvectors.

$\lambda = 1$ : $p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda = 2$ : $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

There are two basic vectors in total; so matrix $A$ is diagonalizable and
\[
P = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix}
\] diagonalizes \( A \).

We know that in this case the matrix \( P^T A P \) will be diagonal with \( \lambda_1, \lambda_2, \ldots, \lambda_n \) as its successive diagonal entries where \( \lambda_i \) is the eigenvalue corresponding to \( \alpha_i \) for \( i = 1, 2, \ldots, n \).

Thus we have:

\[
D = P^T A P = \begin{bmatrix}
1 & 0 \\
0 & 2 \\
\end{bmatrix}
\]

So we obtain:

\[
A^{10} = P D^{10} P^{-1} = \begin{bmatrix}
1 & 0 \\
1 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 2^{10} \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-1 & 2 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1024 & 2048 \\
\end{bmatrix}
\]

\[
A^{10} = \begin{bmatrix}
1 & 0 \\
-1024 & 2048 \\
\end{bmatrix}
\]

20) Compute stated power:

\[ A = \begin{bmatrix}
1 & -2 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix} \]

(a) \( A^{-2301} \)

We know that:

\[ A^k = P D^k P^{-1} \]

Where \( D \) is a diagonal matrix and \( k \) is a positive integer. In our case, \( k \) is negative. We can write:

\[ A^{-2301} = \left( A^{-1} \right)^{2301} \]

Now we can use the formula stated above for:

\[ B = A^{-1} = \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \]

let us find the diagonal matrix.
The given matrix is an upper triangular matrix. Thus, by inspection, we find the eigenvalues and the corresponding eigen vectors.

\( \lambda = -1; \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \lambda = 1; \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

so, the matrix \( B \) is diagonalizable and

\[
\mathbf{p} = \begin{bmatrix} -4 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

diagonalizes \( B \).

We know that in this case, the matrix \( \mathbf{p}^T B \mathbf{p} \) will be diagonal with \( \lambda_1, \lambda_2, \ldots, \lambda_n \) as its successive diagonal entries, where \( \lambda_n \) is the eigenvalue corresponding to \( \mathbf{p}_n \), for \( i = 1, 2, \ldots, n \).

So we obtain:

\[
B = \mathbf{p} \mathbf{D} \mathbf{p}^T = \begin{bmatrix} -4 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 4 \end{bmatrix}
\]

Thus:

\[
\mathbf{A} = \begin{bmatrix} 3 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

\[
\mathbf{A} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]
1) Find the characteristic equation of the given symmetric matrix, and then by inspection determine the dimensions of the eigenspaces:

\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\]

The characteristic equation of matrix A is

\[
\det (\lambda I - A) = \det \begin{bmatrix}
\lambda - 1 & -2 \\
-2 & \lambda - 4
\end{bmatrix} = (\lambda - 1)(\lambda - 4) - 4
\]

\[
\Rightarrow (\lambda - 1)(\lambda - 4) - 4 = 0
\]

\[
\Rightarrow \lambda^2 - 5\lambda + 4 = 0
\]

\[
\Rightarrow \lambda(\lambda - 5) = 0
\]

Thus, the eigenvalues of A are \( \lambda_1 = 0 \) and \( \lambda_2 = 5 \). So, there are 2 eigen spaces of A.

As we know, the dimension of an eigenspace is the nullity \( (\lambda I - A) \) corresponding to \( \lambda \).

So, if \( \lambda = 0 \) then

\[
(\lambda I - A) = \begin{bmatrix}
1 & -2 \\
-2 & 4
\end{bmatrix}
\]

Reducing to row echelon form has the form \[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\]

Thus, the dimension of the eigenspace of A corresponding to \( \lambda = 0 \) is equal to

nullity \( (\lambda I - A) \) = \( 2 - \text{rank}(\lambda I - A) \) = 2 - 1 = 1.

Analogously, for \( \lambda = 5 \) we have that

\[
(\lambda I - A) = \begin{bmatrix}
4 & -2 \\
-2 & 1
\end{bmatrix}
\]

Reducing to row echelon form has the form \[
\begin{bmatrix}
1 & -\frac{1}{2} \\
0 & 0
\end{bmatrix}
\]

Thus, the dimension of the eigenspace of A corresponding to \( \lambda = 5 \) is equal to

nullity \( (\lambda I - A) \) = \( 2 - \text{rank}(\lambda I - A) \) = 2 - 1 = 1.

Dimension of eigenspace corresponding to \( \lambda = 0 \): \( 1 \)
Dimension of eigenspace corresponding to \( \lambda = 5 \): \( 1 \)
The characteristic equation of matrix $A$ is:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix} = (\det \begin{bmatrix} \lambda - 2 \\ 1 & \lambda - 2 \\ 0 & 0 & \lambda - 2 \end{bmatrix})^2$$

$$= (\lambda - 2 - 1)^2 = (\lambda - 3)^2(\lambda - 1)^2 = 0.$$ 

Thus, the eigenvalues of $A$ are $\lambda = 3$ and $\lambda = 1$, so there are two eigenvalues of $A$. As we know, the dimension of the eigenspace is nullity $(\lambda I - A)$ corresponding to $\lambda = 3$, so, if $\lambda = 3$, then

$$(\lambda I - A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

reducing to row echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the dimension of eigenspace of $A$ corresponding to $\lambda = 3$ equals to nullity $(\lambda I - A): 4 - \text{rank}(\lambda I - A) = 4 - 2 = 2$.

Analogously, for $\lambda = 1$ we have that

$$(\lambda I - A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

reducing to row echelon form has the form
Thus, the dimension of the eigenspace of $A$ corresponding to $\lambda = 1$ equals to the multiplicity $(\lambda - 1) = 4 - \text{rank}(\lambda - I) = 4 - 2 = 2$.

The characteristic equation is $(\lambda - 3)^2(\lambda - 1)^2 = 0$.

The dimension of the eigenspace of $A$ corresponding to $\lambda = 1$ is 2.

The dimension of the eigenspace of $A$ corresponding to $\lambda = 3$ is 2.

2) Find a matrix $P$ that orthogonally diagonalizes $A$, and determine $P^2 A P$.

Given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The characteristic equation of matrix $A$ is

$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{bmatrix} = (\lambda - 3)^2 + 1$

And we find the following bases for the eigenspaces:

- $\lambda = 1$: $P_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- $\lambda = 3$: $P_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

There are two basis vectors in total, so matrix $A$ is diagonalizable, and

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

diagonalizes $A$. As a check, we should verify

$P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$
4) \[ A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \]

The characteristic equation of matrix \( A \) is
\[
\det (\lambda I - A) = \det \begin{bmatrix} \lambda - 6 & 2 \\ 2 & \lambda - 3 \end{bmatrix} = (\lambda - 6)(\lambda - 3) - 4
\]
\[= (\lambda - 2)(\lambda - 7) = 0 \]

and we find the following bases for the eigenspaces:
\[
\lambda = 2: \quad p_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \lambda = 7: \quad p_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}
\]

There are two basic vectors in total, so matrix \( A \) is diagonalizable and
\[
P = \begin{bmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{bmatrix}
\]
diagonalizes \( A \). As a check we should verify that
\[
P^{-1}AP = \begin{bmatrix} \frac{1}{5} & \frac{4}{15} \\ -\frac{2}{5} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{bmatrix}
\]
\[= \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}
\]

\[
P = \begin{bmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{bmatrix}
\]
\[
P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 6 & 7 \end{bmatrix}
\]

5) \[ A = \begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \]

The characteristic equation of matrix \( A \) is
\[
\det (\lambda I - A) = \det \begin{bmatrix} \lambda + 2 & 0 & 36 \\ 0 & \lambda + 3 & 0 \\ 36 & 0 & \lambda + 23 \end{bmatrix}
\]
\[= (\lambda + 3)(\lambda + 2)(\lambda + 23) + 36(-36)(\lambda + 23)
\]
\[= (\lambda + 3)(\lambda + 2)(\lambda + 23)(\lambda + 50) = 0 \]

and we find the following orthonormal bases for the eigenspaces
\( \lambda = -3 : P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \); \( \lambda = 25 : P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \); \( \lambda = -50 : P_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \)

There are two basis vectors in total, so matrix A is diagonalizable and

\[ P = \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} \]

diagonalizes A.

As a check we should verify that

\[ P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \\ 3/5 & 0 & 4/5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3/5 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} \begin{bmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix} \]

10) Assuming that \( b \neq 0 \), find a matrix that orthogonally diagonalizes

\[ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \]

The characteristic equation of this matrix is

\((\lambda - a)(\lambda - a - b) = 0\)

and we find the following orthonormal bases for the eigenspaces.
\[ \lambda = a - b : \quad p_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \lambda = a + b : \quad p_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \end{bmatrix} \]

There are two basis vectors in total, so matrix \( A \) is diagonalizable and

\[ P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \]

diagonalizes \( A \).

12) (b) Find a matrix \( P \) that orthogonally diagonalizes \( I - vv^T \) if

\[ v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

We have that

\[ A = I - vv^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \]

The characteristic equation of this matrix is

\[ \det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{bmatrix} = (\lambda - 1)^2(\lambda + 1) = 0. \]

Thus, the eigenvalues of \( A \) are \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \).

It can be shown that

\[ u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad and \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

form an orthonormal basis for the eigenspace corresponding to \( \lambda = 1 \).

The eigenspace corresponding to \( \lambda = -1 \) has
Finally using $u_1, u_2$ and $u_3$ as column vectors, we obtain

$$
P = \begin{bmatrix}
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
$$

which orthogonally diagonalizes $A$. 

$$
\begin{bmatrix}
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
$$

$U_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ as an orthonormal basis.