6.3

1) Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$?
   
   (a) $(0, 1), (2, 0)$
   
   For set of vectors $(0, 1), (2, 0)$ we have
   
   $(0, 1) \cdot (2, 0) = 0 \cdot 2 + 1 \cdot 0 = 0$.
   
   \text{Vectors are orthogonal.}

   (b) $\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$
   
   $\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{1}{2} + \frac{1}{2} = 1$.
   
   \text{Vectors are orthogonal.}

   (c) $\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$
   
   $\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{1}{2} - \frac{1}{2} = -1 \neq 0$.
   
   \text{Vectors are not orthogonal.}

   (d) $(0, 0), (0, 1)$
   
   $(0, 0) \cdot (0, 1) = 0 \cdot 0 + 0 \cdot 1 = 0$.
   
   \text{Vectors are orthogonal.}

   Orthogonal vectors: (a), (b), (d)

   Non-orthogonal vectors: (c)

2) Verify that the set of vectors $\{(1, 0), (0,1)\}$ is orthogonal with respect to the inner product $\langle u, v \rangle = 4u_1v_1 + u_2v_2$ on $\mathbb{R}^2$.

   Let's verify that the given set of vectors is orthogonal or not.
   
   For the inner product $\langle u, v \rangle = 4u_1v_1 + u_2v_2$ we have
   
   $\langle (1, 0), (0, 1) \rangle = 4 \cdot 1 \cdot 0 + 1 \cdot 0 = 0$.
So, the vectors are orthogonal.

The vector \((1,0)\) has a norm \(\sqrt{\langle (1,0),(1,0) \rangle} = \sqrt{1^2 + 0^2} = \sqrt{1} = 1\).

The vector \((\frac{1}{2},0)\) has a norm \(\sqrt{\langle \left(\frac{1}{2},0\right),\left(\frac{1}{2},0\right) \rangle} = \sqrt{\frac{1}{4} + 0^2} = \frac{1}{2} \neq 1\).

So, the vector \((1,0)\) has a norm 1.

The vector \((0,1)\) has a norm \(\sqrt{\langle (0,1),(0,1) \rangle} = \sqrt{0^2 + 1^2} = 1\).

So, the vector \((0,1)\) has a norm 1.

Orthogonal set: \((\frac{1}{2},0), (0,1)\)

11) In each part, an orthonormal basis relative to the Euclidean inner product is given. Use Theorem 6.3.1 to find the coordinate vector of \(\mathbf{w}\) with respect to the basis.

\(\mathbf{w} = (3,4)\), \(\mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\), \(\mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\)

\[\langle \mathbf{w}, \mathbf{u}_1 \rangle = \frac{3}{\sqrt{2}} - \frac{4}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\]

\[\langle \mathbf{w}, \mathbf{u}_2 \rangle = \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{7}{\sqrt{2}}\]

Therefore, if \(S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}\) is an orthonormal basis for an inner product space \(V\), and \(\mathbf{u}\) is any vector in \(V\) then \(u = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n\)

So, we have

\[\mathbf{w} = -\frac{1}{\sqrt{2}} \mathbf{u}_1 + 10 \frac{7}{\sqrt{2}} \mathbf{u}_2\]

The coordinate vector of \(\mathbf{w}\) relative to \(S\) is

\((\mathbf{w})_S = (\langle \mathbf{w}, \mathbf{u}_1 \rangle, \langle \mathbf{w}, \mathbf{u}_2 \rangle)\)

\[\therefore (\mathbf{w})_S = \left(\frac{-1}{\sqrt{2}}, \frac{70}{\sqrt{2}}\right) = (-2\sqrt{2}, 5\sqrt{2})\]

C6) \(\mathbf{w} = (-1,0,2)\), \(\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\), \(\mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\), \(\mathbf{u}_2 = \left(\frac{4}{3}, \frac{2}{3}, -\frac{8}{3}\right)\)
\[ \langle \mathbf{w}, \mathbf{u}_1 \rangle = -\frac{2}{3} + \frac{2}{3} \]
\[ \langle \mathbf{w}, \mathbf{u}_2 \rangle = -\frac{2}{3} - \frac{2}{3} \]
\[ \langle \mathbf{w}, \mathbf{u}_3 \rangle = -\frac{2}{3} + \frac{2}{3} \]

Therefore, if \( S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \) is an orthonormal basis for an inner product space \( V \), and \( \mathbf{u} \) is any vector in \( V \) then \( \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \).

So, we have \( \mathbf{w} = \frac{1}{3} \mathbf{v}_1 - \frac{1}{3} \mathbf{v}_2 + \frac{1}{3} \mathbf{v}_3 \).

\[(\mathbf{w})_5 = \left( \frac{1}{3}, 0, \frac{1}{3}, 1, \frac{1}{3} \right) = (0, -2, 1) \]

12) Let \( \mathbb{R}^2 \) have the Euclidean inner product and let \( S = \{\mathbf{w}_1, \mathbf{w}_2\} \) be the orthonormal basis with \( \mathbf{w}_1 = (3/5, 4/5) \) \( \mathbf{w}_2 = (4/5, 3/5) \).

(a) Find the vectors \( \mathbf{u} \) and \( \mathbf{v} \) that have coordinate vectors \((\mathbf{u})_5 = (1, 1)\) and \((\mathbf{v})_5 = (-1, 4)\).

If \( S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\} \) is an orthonormal basis for an inner product space \( V \), and \( \mathbf{u} \) is any vector in \( V \) then
\[ \mathbf{u} = \langle \mathbf{u}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{u}, \mathbf{w}_2 \rangle \mathbf{w}_2 + \cdots + \langle \mathbf{u}, \mathbf{w}_n \rangle \mathbf{w}_n \]

So, \( \mathbf{u} = -1 \left( \frac{3}{5}, -\frac{4}{5} \right) + 1 \left( \frac{4}{5}, \frac{3}{5} \right) = \left( \frac{1}{5}, -\frac{1}{5} \right) \).

\[ \mathbf{v} = 1 \left( \frac{3}{5}, \frac{4}{5} \right) + 4 \left( \frac{4}{5}, \frac{3}{5} \right) = \left( \frac{18}{5}, \frac{25}{5} \right) \]

\[ (\mathbf{u})_5 = \left( \frac{3}{5}, -\frac{4}{5} \right) \]
\[ (\mathbf{v})_5 = \left( \frac{18}{5}, \frac{25}{5} \right) \]

13) Let \( \mathbb{R}^3 \) have the Euclidean inner product and let \( S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \) be the orthonormal basis with \( \mathbf{w}_1 = (0, -3, 6) \) \( \mathbf{w}_2 = (1, 0, 0) \) and \( \mathbf{w}_3 = (0, 4/3, 3/3) \).

If \( S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \) is an orthonormal basis for an inner product space, \( W \), and \( \mathbf{u} \) is any vector in \( W \) then
Find the vectors \( u,v \) and \( w \) that have the coordinate vectors \((u)_s = (-2,1,2)\), \((v)_s = (3,0,-2)\), and \((w)_s = (5,-4,1)\).

\[
\begin{align*}
u &= -2 \cdot (0, -\frac{2}{5}, \frac{4}{5}) + 1 \cdot (1,0,0) + 2 \cdot (0, \frac{8}{5}, \frac{2}{5}) \\
&= \left( 1, \frac{24}{5}, -\frac{2}{5} \right) \\
v &= 3 \cdot \left( 0, -\frac{3}{5}, \frac{4}{5} \right) + 0 \cdot (3,0,0) - 2 \cdot (0, \frac{8}{5}, \frac{2}{5}) \\
&= \left( 0, -\frac{17}{5}, \frac{6}{5} \right) \\
w &= 5 \cdot \left( 0, -\frac{8}{5}, \frac{4}{5} \right) - 4 \cdot (1,0,0) + 1 \cdot (0, \frac{8}{5}, \frac{8}{5}) \\
&= \left( -4, -\frac{11}{5}, \frac{23}{5} \right)
\end{align*}
\]

\[
\begin{array}{c}
u = \left( 1, \frac{24}{5}, -\frac{2}{5} \right) \\
v = \left( 0, -\frac{17}{5}, \frac{6}{5} \right) \\
w = \left( -4, -\frac{11}{5}, \frac{23}{5} \right)
\end{array}
\]

Let \( R^4 \) have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis \( \{u_1, u_2, u_3, u_4\} \) into an orthonormal basis.

\[
\begin{align*}
u_1 &= (0, 2, 1, 0) \\
u_2 &= (1, -1, 0, 0) \\
u_3 &= (1, 2, 0, -1) \\
u_4 &= (1, 0, 0, 1)
\end{align*}
\]

\[
\begin{align*}
v_1 &= \text{proj}_{u_2} u_1 = u_1 - \frac{\langle u_1, v_1 \rangle}{\|v_1\|^2} v_1 \\
&= (1, -1, 0, 0) - \frac{2}{5} \cdot (0, 2, 1, 0) \\
&= (1, -\frac{4}{5}, \frac{2}{5}, 0)
\end{align*}
\]

\[
\begin{align*}
v_3 &= \text{proj}_{u_2} u_3 = u_3 - \frac{\langle u_3, v_3 \rangle}{\|v_3\|^2} v_1 \\
&= (1, 2, 0, -1) - \frac{\frac{2}{5} \cdot (0, 2, 1, 0)}{\|v_3\|^2} \cdot (1, -\frac{4}{5}, \frac{2}{5}, 0) \\
&= (\frac{1}{5}, \frac{1}{5}, -1, -1)
\end{align*}
\]
The norm of these vectors are:
\[ \|v_1\| = \sqrt{5}, \quad \|v_2\| = \sqrt{15}, \quad \|v_3\| = \sqrt{5}, \quad \|v_4\| = \frac{4}{\sqrt{15}}. \]

So, an orthonormal basis for \( p_4 \) is:
\[ q_1 = \frac{v_1}{\|v_1\|}, \quad q_2 = \frac{v_2}{\|v_2\|}, \quad q_3 = \frac{v_3}{\|v_3\|}, \quad q_4 = \frac{v_4}{\|v_4\|}. \]

Let \( \langle \cdot, \cdot \rangle \) have the inner product \( \langle u_1, v \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3 \).
Use the Gram-Schmidt process to transform \( u_1 = (1, 1, 1) \), \( u_2 = (1, 1, 0) \), \( u_3 = (1, 0, 0) \) into an orthonormal basis.

\[ v_1 = u_1 = (1, 1, 1), \]
\[ v_2 = u_2 - \text{Proj}_v v_2 = u_2 - \frac{(u_2, v_2)}{\|v_2\|^2} v_2 = (1, 1, 0) - \frac{2}{3} (1, 1, 1) = \left( \frac{5}{3}, \frac{5}{3}, -\frac{2}{3} \right), \]
\[ v_3 = u_3 - \text{Proj}_{v_1} v_3 - \text{Proj}_{v_2} v_3 = u_3 - \frac{1}{2} v_1 - \frac{1}{2} v_2 = (1, 1, 1) - \frac{1}{2} (1, 1, 1) - \frac{1}{2} (1, 1, 0) = (0, 0, 1), \]
\[ v_4 = u_4 - \text{Proj}_{v_1} v_4 - \text{Proj}_{v_2} v_4 - \text{Proj}_{v_3} v_4 = u_4 - \frac{1}{2} v_1 - \frac{1}{2} v_2 - \frac{1}{2} (0, 0, 1) = (0, 0, 1). \]
\[ \mathbf{u}_3 = \frac{\langle \mathbf{u}_2, \mathbf{v}_2 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2}{\| \mathbf{v}_2 \|^2} \]
\[ = \left( \frac{1}{3}, 0, 0 \right) - \frac{1}{3} \left( \frac{1}{3}, \frac{1}{3}, 1 \right) - \frac{1}{2} \left( 1, 1, 0 \right) \]
\[ = \left( \frac{1}{16}, -\frac{5}{6}, -\frac{1}{\sqrt{3}} \right) \]

The norms of these vectors are
\[ \| \mathbf{v}_1 \| = \sqrt{3}, \quad \| \mathbf{v}_2 \| = \sqrt{\frac{2}{3}}, \quad \| \mathbf{v}_3 \| = \sqrt{\frac{4}{5}} \]

\[ q_1 = \frac{\mathbf{v}_1}{\| \mathbf{v}_1 \|} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \]
\[ q_2 = \frac{\mathbf{v}_2}{\| \mathbf{v}_2 \|} = \sqrt{\frac{2}{3}} \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right) \]
\[ q_3 = \frac{\mathbf{v}_3}{\| \mathbf{v}_3 \|} = \sqrt{\frac{2}{5}} \left( \frac{1}{16}, -\frac{5}{6}, -\frac{1}{\sqrt{3}} \right) \]

21) The subspace of \( \mathbb{R}^3 \) spanned by the vectors \( \mathbf{u}_1 = (4, 5, 0, -3/\sqrt{2}) \) and \( \mathbf{u}_2 = (0, 1, 0) \) is a plane passing through the origin. Express \( \mathbf{w} = (1, 2, 3) \) in the form \( \mathbf{w} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 \), where \( c_1 \) lies in the plane and \( \mathbf{w}_2 \) is perpendicular to the plane.

\[ \mathbf{w} = \text{proj}_{\mathbf{u}_1, \mathbf{u}_2} \mathbf{w} = (c_1, c_2) \mathbf{u}_1 + (c_2, 0, 0) \mathbf{u}_2 \]
\[ = \left( \frac{4}{3}, -\frac{7}{3} \right) \cdot \left( \frac{4}{5}, 0, -\frac{3}{5} \right) + (2) \left( 0, 1, 0 \right) \]
\[ = \left( \frac{4}{5}, 0, -\frac{3}{5} \right) + (0, 2, 0) = \left( -\frac{4}{5}, 2, \frac{3}{5} \right) \]

The component of \( \mathbf{w} \) orthogonal to \( \mathbf{u}_1, \mathbf{u}_2 \) is \( \mathbf{w}_2 = \text{proj}_{\mathbf{u}_1, \mathbf{u}_2} \mathbf{w} = \mathbf{w} - \text{proj}_{\mathbf{u}_1, \mathbf{u}_2} \mathbf{w} \).
23) Let \( p^4 \) have the Euclidean inner product. Express \( w = (-1, 2, 6) \) in the form \( w = w_1 + w_2 \), where \( w_1 \) is in the space \( W \) spanned by \( u_1 = (-1, 0, 1, 2) \) and \( u_2 = (0, 1, 0, 1) \), and \( w_2 \) is orthogonal to \( w_1 \).

\[
P = u_2 - \frac{u_1 \cdot u_2}{\|u_1\|^2} u_1.
\]

\[
= (0, 1, 0, 1) - \frac{2}{6} (-1, 0, 1, 2)
\]

\[
P = (1/3, 1, -1/3, 1/3).
\]

Now, \( w_1 = \frac{w \cdot u_1}{\|u_1\|^2} u_1 + \frac{w \cdot u_2}{\|u_2\|^2} u_2 \)

\[
= \frac{2}{6} (-1, 0, 1, 2) + \frac{\sqrt{2}}{6} \left( \frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{3} \right)
\]

\[
= (-\frac{7}{6}, 0, \frac{7}{6}, \frac{14}{6}) + (\frac{1}{2}, -\frac{1}{6}, \frac{1}{2}, \frac{1}{6})
\]

\[
= w_1 = (-\frac{7}{6}, -\frac{1}{6}, \frac{8}{6}, \frac{9}{6})
\]

Now, \( w_2 = w - w_1 \)

\[
w_2 = w_1 - w_1
\]

\[
= (-1, 2, 6, 0) - (-5/4, -1/4, 5/4, 9/4)
\]

\[
= (-1/4, 9/4, 19/4, -9/4)
\]

\[
w_1 = (\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4})
\]

\[
w_2 = (-\frac{1}{4}, \frac{9}{4}, \frac{19}{4}, -\frac{9}{4})
\]
6.4

1) Find the normal system associated with the given linear system.

(a) \[
\begin{bmatrix}
1 & -1 \\
2 & 3 \\
4 & 5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix}
\]

In this system, the given linear system can be written as \(Ax = b\). By definition, the normal system associated with \(Ax = b\) is given as \(A^TAx = A^Tb\).

where \(A^T\) denotes the matrix transpose obtained by exchanging \(A\)'s rows and columns.

So, we have

\[
A^T = \begin{bmatrix}
1 & 2 & 4 \\
-1 & 3 & 5
\end{bmatrix}
\]

Now, we are going to calculate the matrix \(A^TA\) and the vector \(A^Tb\). \n
\[
A^TA = \begin{bmatrix}
1 & 2 & 4 \\
-1 & 3 & 5
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
2 & 3 \\
4 & 5
\end{bmatrix} = \begin{bmatrix}
21 & 25 \\
25 & 35
\end{bmatrix}
\]

And

\[
A^Tb = \begin{bmatrix}
1 & 2 & 4 \\
-1 & 3 & 5
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix} = \begin{bmatrix}
20 \\
20
\end{bmatrix}
\]

Thus, the correct answer is

\[
\begin{bmatrix}
21 & 25 \\
25 & 35
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
20 \\
20
\end{bmatrix}
\]

5) Find the least square solutions of the linear system \(Ax = b\) and find the orthogonal projection of \(b\) onto the (column) space of \(A\).

(a) \[A = \begin{bmatrix}
1 & 1 \\
-1 & 1 \\
-1 & 2
\end{bmatrix}, \quad b = \begin{bmatrix}
7 \\
0 \\
-7
\end{bmatrix}\]
By definition, the normal system associated with $Ax = b$ is given as

$$A^T Ax = A^T b.$$ 

$$A^T = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 14 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} = 16 - 4 = 12 \neq 0$$

$$x_1 = \frac{14 - 2}{3 - 2} = \frac{12}{1} = 12$$

$$x_2 = \frac{3 - 2}{-2 - 6} = \frac{-5}{-8} = \frac{5}{8}$$

Let $w$ denote the column space of $A$ and $proj_w b$ be the orthogonal projection of $b$ on $w$.

If $x$ is any solution of the normal system then $proj_w b = Ax$.

$$proj_w b = Ax \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 12 \\ -5 \\ -9 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \\ -4 \end{bmatrix}$$

$$x_1 = \frac{12}{8} = \frac{3}{2}, \quad x_2 = \frac{5}{2}$$

$$y_1 = \frac{14}{12}, \quad y_2 = \frac{-5}{12}, \quad z = \frac{-4}{12}$$
\(1.0\) Let \( w \) be the plane with equation \( 5x - 3y + z = 0 \).

(a) Find a basis for \( w \).
\[
\begin{align*}
  5x - 3y + z &= 0 \\
  5x &= 3y - z \\
  x &= \frac{3}{5}y - \frac{1}{5}z.
\end{align*}
\]
It follows that \( y \) and \( z \) are free variables. Consider 2 vectors, \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in \( w \):
\[
\mathbf{v}_1 = (0.6, 1.8, 0) \quad \text{and} \quad \mathbf{v}_2 = (0.5, 0.2, 1) (-1/5, 0, 1)
\]
where \( x \) components. \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are defined from
\[
x = \frac{3}{5}y - \frac{1}{5}z = 0.6y - 0.2z
\]
Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent. 

If \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly dependent vectors then there are \( \alpha \) and \( \beta \) such that \( \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{0} \).
Writing the last \( \mathbf{v}_2 \) by components \( g=0 \):
\[
\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = (\alpha (0.6) + \beta (1), \alpha (1.8) + \beta (0), \alpha, \beta) = (0, 0, 0)
\]
It follows that \( \alpha = 0 \) and \( \beta = 0 \).
Hence \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly dependent, and these vectors form a basis for \( w \).

\[
\begin{bmatrix}
  0.6 & -0.2 \\
  1 & 0 \\
  0 & 1
\end{bmatrix}
\]

(b) Use the matrix obtained in (a) to find the orthogonal projection of \( w \).
\[
5x - 3y + z = 0.
\]
\[
P = A (A^T A)^{-1} A^T
\]
Since the coefficient of \( x \) is non-zero, we can express \( x \) from the given equation. We have
\[
x = \mathbf{v}_1 + \mathbf{v}_2.
\]
It follows that \( y \) and \( z \) are free variables. Consider 2 vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in \( \mathbf{w} \)
\[
\mathbf{v}_1 = (3, 5, 0) \quad \text{and} \quad \mathbf{v}_2 = (-3, 0, 5), \quad \text{where}
\]
the component of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are defined for \( \mathbf{w} = \frac{y}{2} - \frac{z}{2} \).
\[
A = \begin{bmatrix}
8 & -1 \\
5 & 0 \\
0 & 5
\end{bmatrix}
\]
\[
A^T A = \begin{bmatrix}
3 & 5 & 0 \\
-1 & 0 & 5 \\
0 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
3 & -1 \\
5 & 0 \\
0 & 5
\end{bmatrix} = \begin{bmatrix}
34 & -3 \\
-3 & 26
\end{bmatrix}
\]
\[
(A^T A)^{-1} = \frac{1}{\text{det}(A^T A)} \begin{bmatrix}
26 & -3 \\
-3 & 34
\end{bmatrix} = 875 \begin{bmatrix}
26 & -3 \\
-3 & 34
\end{bmatrix}
\]
\[
P = A (A^T A)^{-1} A^T = \begin{bmatrix}
3 & -1 \\
5 & 0 \\
0 & 5
\end{bmatrix} \cdot \begin{bmatrix}
26 & -3 \\
-3 & 34
\end{bmatrix} \cdot \begin{bmatrix}
3 & -1 \\
5 & 0 \\
0 & 5
\end{bmatrix}
\]
\[
= \frac{1}{875} \begin{bmatrix}
3 & -1 \\
5 & 0 \\
0 & 5
\end{bmatrix} \begin{bmatrix}
75 & 150 & 15 \\
-25 & 15 & 170
\end{bmatrix}
\]
\[
= \frac{1}{875} \begin{bmatrix}
250 & 375 & -125 \\
375 & 650 & 75 \\
-125 & 95 & 850
\end{bmatrix} = \frac{1}{85} \begin{bmatrix}
10 & 15 & -5 \\
15 & 26 & 3 \\
-5 & 8 & 34
\end{bmatrix}
\]

12) In \( \mathbb{R}^3 \) consider the line \( L \) given by the equations
\( (x = t, y = t, z = t) \) and the line \( m \) given by the equation
\( (x = -2, y = -2, z = -2) \). Let \( P \) be a point on \( L \) and let \( Q \)
be as point on on. Find the values of \( t \) and \( s \) that minimize the distance between the lines by minimizing the square distance \( ||P-Q||^2 \). Let \( P \) and \( Q \) denote respectively the point on \( L_1 \) that corresponds to \( t \) and the point on that corresponds to \( s \). Denote by \( u \), the leading vectors of the line \( L_1 \) by and by \( v \) the leading vectors of line \( L_2 \). \( U = (1, 1, 1) \), \( V = (3, 2, 1) \).

\[
A = \begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 1
\end{bmatrix}
\Rightarrow A^T A x = A^T b
\]

\[
A^T = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1
\end{bmatrix}
\]

\[
A^T A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
3 & 4 \\
4 & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 4 \\
4 & 6
\end{bmatrix}
\begin{bmatrix}
t \\
s
\end{bmatrix} = 0
\]

since \( \begin{bmatrix}
3 & 4 \\
4 & 6
\end{bmatrix} \begin{bmatrix}
t \\
s
\end{bmatrix} = 16t - 16 = 2 \neq 0 \) it has only trivial sol.

Thus the target parameters \( t \) and \( s \) that minimize the distance between \( 2 \) and \( m \) are \( t = 0 \), \( s = 0 \).

\[
\begin{bmatrix}
t = 0 \\
s = 0
\end{bmatrix}
\]

b) The relationship between the current \( I \) through a resistor and the voltage drop \( V \) across it is given by Ohm's law \( V = IR \). Successive experiments are performed in which a known current (measured in amps) is passed through a resistor \( R \) of unknown resistance \( R \) and the voltage drop (measured in volts) is measured. This result in the \((I, V)\) data \((0.4, 1.1), (0.2, 0.2), (0.4, 0.2), (0.5, 0.5)\). The data,
is assumed to have measurement errors that prevent it from following Ohm's law precisely.

(i) Find the least square solution of this system and interpret your result.

Let \( \mathbf{I} \) denote the 5x1 matrix of the given system that can be considered as a vector in \( \mathbb{R}^5 \); and \( \mathbf{V} \) represent voltage.

\[
\mathbf{I} = \begin{bmatrix}
0.1 \\
0.2 \\
0.3 \\
0.4 \\
0.5 \\
\end{bmatrix}
\]

\[
\mathbf{V} = \begin{bmatrix}
1 \\
2.1 \\
2.9 \\
4.2 \\
5.1 \\
\end{bmatrix}
\]

From Ohm's law, \( \mathbf{I} = \mathbf{V} \mathbf{x} \).

\[
\Rightarrow \mathbf{V}^T \mathbf{x} = \mathbf{I}^T \
\]

\[
\Rightarrow \begin{bmatrix}
0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\
\end{bmatrix} \begin{bmatrix}
0.1 \\
0.2 \\
0.3 \\
0.4 \\
0.5 \\
\end{bmatrix} \mathbf{V} = \begin{bmatrix}
500 \\
500 \end{bmatrix}
\]

\[
\Rightarrow 0.1 \times 5 = 0.5 \
\]

\[
\Rightarrow 0.2 \times 5 = 1 \
\]

\[
\Rightarrow 0.3 \times 5 = 1.5 \
\]

\[
\Rightarrow 0.4 \times 5 = 2 \
\]

\[
\Rightarrow 0.5 \times 5 = 2.5 \
\]

Thus, \( \frac{0.5 \times x = 5.62}{0.5} = 10.21 \approx 10 \)

Thus, \( R = 10 \) is the best O(n) approximation in \( \mathbb{R}^5 \) for the experiment device over all possible values of the coefficient of proportionality between the current \( I \) through a resistor and the voltage drop \( V \) across it.

\[
R = 10 \ \text{ohms}.
\]
6.5 (A)

1) Find the coordinate vector for \( \mathbf{w} \) relative to the basis \( \mathbf{S} = \{\mathbf{u}_1, \mathbf{u}_2\} \) for \( \mathbb{R}^2 \).

(b) \( \mathbf{u}_1 = (2, -4), \mathbf{u}_2 = (3, 8), \mathbf{w} = (3, 11) \)

The non-zero basis vectors for the new basis \( \mathbf{S} \),
\( \mathbf{u}_1 = (2, 3) \) and \( \mathbf{u}_2 = (3, 8) \) are the coordinate vectors relative to the basis vectors \((1, 0)\) and \((0, 1)\) of the old basis \( \mathbf{A} \).

Then the transition matrix from the new basis \( \mathbf{S} \) to the old basis \( \mathbf{A} \)

\[
\mathbf{P} = \begin{bmatrix} \mathbf{u}_1 | \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix}
\]

\( [\mathbf{w}]_A = (1, 1) \) is the coordinate vector for \( \mathbf{w} \) relative to the old basis.

So, the coordinate vector for \( \mathbf{w} \) relative to the new basis \( \mathbf{S} \) will be

\[
[\mathbf{w}]_S = \mathbf{P}^{-1} [\mathbf{w}]_A = \frac{1}{2 \times 8} \begin{bmatrix} 8 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2 \times 8} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 6/28 \end{bmatrix}
\]

\[
[\mathbf{w}]_S = \begin{bmatrix} 5/28 \\ 6/28 \end{bmatrix}
\]

3) Find the coordinate vector for \( \mathbf{p} \) relative to \( \mathbf{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \).

(a) \( \mathbf{p} = 4 - 3x + x^2, \mathbf{p}_1 = 1, \mathbf{p}_2 = x, \mathbf{p}_3 = x^2 \)

Suppose \( \mathbf{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \) is a basis for a space \( \mathbf{P}_2 \) of polynomials of degree at most 2, where

\( \mathbf{p}_1 = 1, \mathbf{p}_2 = x, \mathbf{p}_3 = x^2 \)

A polynomial \( \mathbf{p} = 4 - 3x + x^2 \) in \( \mathbf{P}_2 \) can be expressed in terms of the basis \( \mathbf{S} \) as

\[
\mathbf{p} = 4 \mathbf{p}_1 - 3 \mathbf{p}_2 + \mathbf{p}_3.
\]

Thus, numbers 4, -3 and 1 are called the coordinates of \( \mathbf{p} \) relative to \( \mathbf{S} \). The vector \((4, -3, 1)\) constructed from
these coordinate vector $P$ relative to $S'$.

$$[P]_{S'} = (4, -3, 1)$$

4) Find the coordinate vector for $A$ relative to $S': A_1, A_2, A_3, A_4$

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in $P'$ can be associated with the matrices $A_1, A_2, A_3, A_4$. The vectors $v_1, v_2, v_3, v_4$ are linearly independent. Hence, these vectors can define a basis in $P'$. $S' = \{v_1, v_2, v_3, v_4\}$

The transition matrix from this new basis $S'$ to the standard basis $B = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ in $R^4$ is

$$p = [v_1 | v_2 | v_3 | v_4] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse matrix is

$$p^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, the matrix $A = [2, 0; -1, 3]$ can be represented by the vector $[v_1]_{S'} = (2, 0, -1, 3)$ in $P'$, whose coordinate vector relative to basis $S'$ is...
\[
\begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
-1 \\
3
\end{bmatrix}
\]

In coordinate vector \( v \):
\[
\begin{bmatrix}
3 \\
4
\end{bmatrix}
\]

6) Consider the bases \( B = \{u_1, u_2\} \) and \( B' = \{v_1, v_2\} \) for \( \mathbb{R}^2 \), where

\[
u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}
\]

(a) Find the transition matrix from \( B' \) to \( B \).

The transition matrix \( P \) from \( B' = \{v_1, v_2\} \) to \( B = \{u_1, u_2\} \) can be expressed in terms of its column vectors:

\[
P = \begin{bmatrix}
[v_1]_B \\
[v_2]_B
\end{bmatrix}
\]

\[
v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}
\]

Then,

\[
[v_1]_B = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2u_1 + u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

and

\[
[v_2]_B = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3u_1 + u_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}
\]

The transition matrix from the basis \( B' \) to the basis \( B \) is then

\[
P = \begin{bmatrix}
[v_1]_B \\
[v_2]_B
\end{bmatrix} = \begin{bmatrix}
2 & -3 \\
1 & 4
\end{bmatrix}
\]

Transition matrix