No:	5.5
Total	08

1A

5.5

2) Express the product Ax as a linear combination of the column vectors of A.

(a)

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If $c_1, c_2, \ldots c_n$ denote the column vectors of A, and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then the product Ax can be expressed as a linear combination of these column vectors with coefficients from x; that is,

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n$$

Therefore

$$\boxed{\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

5) Determine weather b is in the column space of A, and if so, express b as a linear combination of the column vectors of A.

(a)

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}; \quad b = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A.

Therefor we find the solution of $Ax = b$, where

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \text{ and } b = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

It is evident that,

$$x_1 = 1, \; x_2 = -1$$

solves the system of linear equations. Since the system is consistent, b is in the column space of A. Moreover,

$$b = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$\boxed{\text{b is in the column space of A and } b = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}}$$

4) Suppose that $x_1 = -1$, $x_3 = 2$, $x_3 = 4$, ~~and~~ $x_4 = -3$ is a solution of a nonhomogeneous linear system $Ax = b$ and that the solution set of the homogeneous system $Ax = 0$ is given by the formulas

$$x_1 = -3r + 4s$$
$$x_2 = r - s$$
$$x_3 = r$$
$$x_4 = s$$

(a) Find the vector form of the general solution of $Ax = 0$.

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

(b) Find the vector form of the general solution of

If x_0 denotes any single solution of a consistent linear system $Ax = b$ and if $v_1, v_2, \ldots, v_k$ forms a basis for the null space of A - that is, the solution space of the homogeneous system $Ax=0$, then every solution of $Ax=b$ can be expressed in the form

$$x = x_0 + c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

and, conversely, for all choices of scalars $c_1, c_2, \ldots, c$ the vector x in this formula is solution of $Ax=b$

We have that

$$x_0 = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}$$

and every solution of $Ax=0$ can be written as

$$r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the general solution of $Ax=b$ is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

6) Find a basis for the nullspace of A.

(a)

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

The nullspace of A is the solution space of the homogeneous system

2B

$$x_1 - x_2 + 3x_3 = 0$$
$$5x_1 - 4x_2 - 4x_3 = 0$$
$$7x_1 - 6x_2 + 2x_3 = 0$$

We solve the given system of linear equations by using Gauss-Jordan elimination. The augmented matrix for the system is

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{bmatrix}$$

After reduction we obtain

$$\begin{bmatrix} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad x_1 \quad -16x_3 = 0$$
$$x_2 - 19x_3 = 0$$

$$\Rightarrow \quad x_1 = 16x_3 \quad \& \quad x_2 = 19x_3$$
$$\Rightarrow \quad x_1 = 16s, \quad x_2 = 19s, \quad x_3 = s$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

$\therefore$ vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms the basis for solution space.

Consequently it also forms a basis for the null space of A.

$$\therefore \boxed{\text{basis } \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}}$$

7) In each part, a matrix in row-echelon form is given. By inspection, find bases for the row and column spaces of A.

(a)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The following theorem makes it possible to find basis for the row and column spaces of a matrix in row-echelon form by inspection

If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of r.

the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form. From this theorem, the vectors

$$a_1 = [1 \quad 0 \quad 2]$$
$$a_2 = [0 \quad 0 \quad 1]$$

form a basis for the row space of A, and the vectors

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad c_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of A.

$$\boxed{\begin{array}{ll} \text{Row basis} & [1 \quad 0 \quad 2], [0, 0, 1] \\ \text{Column basis} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array}}$$

3B

11) Find a basis for the subspace of R^4 spanned by the given vectors.

(a) $(1,1,-4,-3)$, $(2,0,2,-2)$, $(2,-1,3,2)$

Except for a variation in notation, the space spanned by these vectors in row space of matrix.

$$\begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$$

Reducing this to row echelon form, we obtain

$$\begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

The following theorem makes it possible to find the basis for the row space of a matrix in row-echelon form by inspection.

"If a matrix R is in row echelon form, then the row vectors with leading 1's (the nonzero row vectors) form a basis for the row space of R"

∴ The matrix $\begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$

is in row echelon form. From this theorem the vectors

$$\boxed{(1,1,-4,-3),\ (0,1,-5,-2),\ (0,0,1,-1/2)}$$

12) Find a subset of the vectors that forms basis for the space spanned by the vectors, then express each vector that is not in the basis as a linear combination of the basis vectors.

4A

(a) $v_1 = (1,0,1,1)$, $v_2 = (-3,3,7,1)$, $v_3 = (-1,3,9,3)$, $v_4 = (-5,3,5,-1)$

We begin by constructing a matrix that has v_1, v_2, v_3, v_4 as its column vectors

$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$$

Reducing matrix to row-echelon form and denoting column vectors of resulting matrix by w_1, w_2, w_3, w_4 yields

$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's occur in columns 1 & 2, so by this theorem

$$\{w_1, w_2\}$$

is a basis for the column space of the matrix in reduced row-echelon form, and consequently,

$$\{v_1, v_2\}$$

is a basis for the column space of

$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$$

By inspection

$$v_3 = 2v_1 + v_2$$

& $v_4 = -2v_1 + v_2$

4B

v_1, v_2 forms the basis.

$v_3 = 2v_1 + v_2$

$v_4 = -2v_1 + v_2$

[redacted]

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1A

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(2) Find the rank and nullity of the matrix, then verify that the values obtained satisfy Formula (4) of the dimension theorem.

(a)
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are two non zero rows, the row space and column space are both two dimensiona so rank (A) = 2. To find the nullity of A, we mus find the dimension of the solution space of linear system $Ax = 0$. This system can be solved by reducin the augmented matrix to reduced row echelon form The resulting matrix will be identical to ~~above~~ ~~shown~~ matrix shown above.

$$x_1 - 16x_3 = 0 \quad \text{or} \quad x_1 = 16x_3$$
$$x_2 - 19x_3 = 0 \qquad\quad x_2 = 19x_3$$

$$\therefore \quad x_1 = 16r$$
$$x_2 = 19r$$
$$x_3 = r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

Because the one vector on the right side forms a basis for the solution space, nullity (A) = 1

$$\therefore \underset{\substack{\uparrow \\ \text{Rank}}}{2} + \underset{\substack{\uparrow \\ \text{Nullity}}}{1} = 3$$

$$\boxed{\begin{array}{l} \text{rank}(A) = 2 \\ \text{nullity}(A) = 1 \end{array}}$$

(e)

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 10 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad x_1 + 4x_4 + 10x_5 = 0$$
$$x_2 + x_5 = 0$$
$$x_3 - x_5 = 0$$

$$\Rightarrow \quad x_1 = -4x_4 - 10x_5$$
$$x_2 = -x_5$$
$$x_3 = x_5$$

$\Rightarrow x_1 = -4r - 10t \; ; \; x_2 = -t \; ; \; x_3 = t \; ; \; x_4 = r \; ; \; x_5 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -10 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{\begin{array}{l} \text{rank}(A) = 3 \\ \text{nullity}(A) = 2 \end{array}}$$

4) In each part, use the information in the table to find the dimension of the row space, column space and nullspace of A, and the nullspace of A^T

(a) size of A 3×3

Rank(A) 3

The rank of A is the common dimension of the row space and column space of matrix A, therefore:

$\dim(\text{row space of } A) = 3,$

$\dim(\text{column space of } A) = 3.$

If A is any matrix, then $\operatorname{rank}(A) = \operatorname{rank}(A^T)$, hence in our case, $\operatorname{rank}(A^T) = 3$. Applying the theorem:

If A is a matrix with n columns then

$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$

to A and A^T yields

$\operatorname{nullity}(A) = 3 - 3 = 0$

$\operatorname{nullity}(A^T) = 3 - 3 = 0$

Row space of $A = 3$
Column space of $A = 3$
Null space of $A = 0$
~~Null space of $A^T = 0$~~

5) In each part, find the largest possible value for the ran of A and the smallest possible value for the nullity of A.

(a) A is 4×4

For any $m \times n$ matrix A $\operatorname{rank}(A) \leq \min(m, n)$.
Hence the largest possible value for the rank of A is the smallest possible value for the nullity of A is

$4 - \max(\operatorname{rank}(A)) = 0$

2B

$\boxed{\begin{array}{l}\text{Largest possible value of rank : 4}\\ \text{Smallest possible value for nullity = 0}\end{array}}$

(b) A is 3×5

For any $m \times n$ matrix A $\text{rank}(A) \le \min(m, n)$.
So, the largest possible value for the rank of A is $\min(3, 5) = 3$.

The smallest possible value for the nullity of A is

$5 - \max(\text{rank}(A))$

$5 - 3 = 2$

$\boxed{\begin{array}{l}\text{Largest possible value of rank = 3}\\ \text{Smallest possible value for nullity : 2}\end{array}}$

7) In each part, use the information in the table to determine whether the linear system $Ax = b$ is consistent. If so, state the number of parameters in its general solution

(a) size of A : 3×3

$\text{Rank}(A) = 3$

$\text{Rank}[A \mid b] = 3$

Since the coefficient matrix A and the augmented matrix $[A \mid b]$ have same rank, the linear system $Ax = b$ is consistent, and the general solution of the system ~~is consistent~~ contains

$3 - \text{rank}(A) = 3 - 3 = 0$ parameters

$\boxed{\begin{array}{c}\text{Linear system is consistent.}\\ \text{No. of Parameters in general solution: 0}\end{array}}$

(c) Size of A: 5×9

$\text{rank}(A)$ 2, $\text{Rank}[A \mid b]$ 3

Since the coefficient matrix A and the augmented matrix [A | b] do not have the same rank, the linear system $Ax=b$ is

$\boxed{\text{not consistent}}$

13) Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1 or 2? If so, find those values.

We can reduce given matrix as shown below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

As we can see, this matrix has rank 2 if and only if $r=2$ and $s=1$.

There are no vales of r and s for which the given matrix has rank 1.

$$\boxed{\begin{array}{l} \text{rank 2: } r=2,\ s=1 \\ \text{rank 1: no such values} \end{array}}$$