1. (10 pts) Find values of $a$ and $b$ such that the polynomial $p(x) = ax^4 + bx$ has a horizontal tangent at the point (1,3).

Answer: $a = -1, b = 4$

2. (20 pts) For which values of $a$ and $b$ does the following linear system

\[
\begin{align*}
ax_1 + x_2 + x_3 &= 4 \\
x_1 + x_2 &= 6 \\
x_1 + x_3 &= b
\end{align*}
\]

a. have one solution? Answer: $a \neq 2$
b. have no solutions? Answer: $a = 2, b \neq -2$
c. have infinitely many solutions? Answer: $a = 2, b = -2$

3. (10 pts) Find the following determinant

\[
\begin{vmatrix}
5 & 31 & 0 & 0 & 8 \\
2 & 8 & 0 & 0 & 7 \\
75 & -87 & 1 & 86 & -21 \\
0 & 1 & 0 & 0 & 0 \\
38 & -47 & 0 & 1 & 92 \\
\end{vmatrix}
\]

Answer: 19

4. (20 pts) Find the inverses of the following matrices

\[
\begin{pmatrix}
2 & 4 \\
1 & 3
\end{pmatrix}^{-1} = \begin{pmatrix}
1.5 & -2 \\
-5 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 4 & 0 \\
1 & 3 & 0 \\
0 & 0 & 5
\end{pmatrix}^{-1} = \begin{pmatrix}
1.5 & -2 & 0 \\
-5 & 1 & 0 \\
0 & 0 & .2
\end{pmatrix}
\]

5. (20 pts) A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sends $e_1$ to $2e_1 - e_2$, $e_2$ to $-3e_1$, and $e_3$
to $5e_3$. What is the standard matrix of $T$? What is the rank of $T$?

Answer: $[T] = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. The rank of $[T]$ is 3.

6. (30 pts) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 12 & 15 \\ 1 & 2 & 4 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 3 & 6 & 10 & 30 & 0 & 0 & 0 \\ 1 & 2 & 3 & 9 & 0 & 0 & 0 \end{pmatrix}$$

a. Find the rank and nullity of $A$ and of $A^T$.
b. Find bases for the row space and column space of $A$.
c. Find bases for $N=(\text{null space of } A)$ and for $N^\perp$.

Answer: The rref of $A$ is $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Thus $\text{rank}(A) = \text{rank}(A^T) = 3$. By the dimension theorem, $\text{rank}(A) + \text{nullity}(A) = 7$, thus nullity of $A$ is 4. Similarly, $\text{rank}(A^T) + \text{nullity}(A^T) = 5$, and thus nullity of $A^T$ is 2.

A basis for the rowspace($A$) is $\{(1,2,0,0,0,0,0), (0,0,1,3,0,0,0), (0,0,0,0,1,4,5)\}$.

A basis for the columnspace($A$) is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 10 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

A basis for the nullspace $N$ of $A$ is $\{(-2,1,0,0,0,0,0), (0,0,-3,1,0,0,0), (0,0,0,0,0,-4,1,0), (0,0,0,0,0,0,0)\}$.

Since $N = \text{rowspace}(A)^\perp$, $N^\perp$ is the same as $(\text{rowspace}(A)^\perp)^\perp = \text{rowspace}(A)$. Thus a basis for it is $\{(1,2,0,0,0,0,0), (0,0,1,3,0,0,0), (0,0,0,0,1,4,5)\}$.

7. (20 pts) (a) What is the transition matrix $P$ from the basis

$$B' = \{u'_1 = 2e_1 - e_2, u'_2 = -3e_1\}$$

to the standard basis $B = \{e_1, e_2\}$?
(b) If \([u]_B = [1, 2]\) and \([v]_{B'} = [3, 4]\), find \([u]_{B'}\) and \([v]_B\).

Answer: \(P = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}\). \([u]_{B'} = [-2, -5/3]\), and \([v]_B = [-6, -3]\).

8. (20 pts) (a) Find an orthonormal basis for \(\mathbb{R}^2\) by applying the Gram-Schmidt orthonormalization process to the basis \(\{(3, 4), (1, 1)\}\).

Answer: \(v_1 = (3, 4)\), and \(u_1 = v_1/\|v_1\| = (3/5, 4/5)\).

\(v_2 = (1, 1) - \langle (3/5, 4/5), (1, 1) \rangle (3/5, 4/5) = (4/25, -3/25)\) and so \(u_2 = v_2/\|v_2\| = (4/5, -3/5)\).

(b) Find the inner product between the vectors \((3, 4)\) and \((1, 1)\) for the inner product generated by \(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\).

Answer:

\[
\langle (3, 4), (1, 1) \rangle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = (3, 8) \cdot (1, 2) = 19.
\]

9. (20 pts) Find the eigenvalues of the matrix

\[
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.
\]

and indicate the algebraic and geometric multiplicity for each eigenvalue you found.

Answer: \[
\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 3 & \lambda - 3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -1 \\ 3 & \lambda - 3 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 0 \\ \lambda & -1 \end{vmatrix} = \lambda [\lambda(\lambda - 3) + 3] - 1
\]

\[
= \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3
\]

Thus the only eigenvalue is 1 and its algebraic multiplicity is 3. Its geometric multiplicity is

\[
\text{nullity}\left(1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}\right) = \text{nullity}\left(\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix}\right)
\]
Putting this into reduced row-echelon form, this equals

\[
\text{nullity}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 3 - \text{rank}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 1. \text{ Thus the geometric multiplicity is 1.}
\]

10. (30 pts) Find an orthogonal matrix \( P \) that diagonalizes \( B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{pmatrix} \) and find \( P^T BP \).

Answer: The characteristic polynomial is

\[
\begin{vmatrix} 
\lambda - 6 & 0 & 0 \\
0 & \lambda - 2 & -3 \\
0 & -3 & \lambda - 4 \\
\end{vmatrix} = (\lambda - 6) [(\lambda - 2)(\lambda - 4) - 9] = 
\]

\[(\lambda - 6)(\lambda - 3 + \sqrt{10})(\lambda - 3 - \sqrt{10}).\]

To find each eigenspace, we calculate the nullspace of \((\lambda I - B)\). For \(\lambda = 6\), we want the nullspace of the matrix

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -3 & 2 \end{pmatrix},
\]

which is spanned by \((1, 0, 0)\). For \(\lambda = 3 + \sqrt{10}\), we want the nullspace of the matrix

\[
\begin{pmatrix} -3 + \sqrt{10} & 0 & 0 \\ 0 & 1 + \sqrt{10} & -3 \\ 0 & -3 & -1 + \sqrt{10} \end{pmatrix},
\]

which is spanned by \((0, 3, 1 + \sqrt{10})\). A unit vector in this direction is \((0, \frac{3}{\sqrt{20 + 2\sqrt{10}}}, \frac{1 + \sqrt{10}}{\sqrt{20 + 2\sqrt{10}}})\). Similarly, for \(\lambda = 3 - \sqrt{10}\) the easy eigenvector to write down is \((0, 3, 1 - \sqrt{10})\) which, when normalized to unit length becomes \((0, \frac{3}{\sqrt{20 - 2\sqrt{10}}}, \frac{1 - \sqrt{10}}{\sqrt{20 - 2\sqrt{10}}})\). Thus the orthogonal matrix that diagonalizes \(B\) is

\[
P = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{3}{\sqrt{20 + 2\sqrt{10}}} & \frac{1 + \sqrt{10}}{\sqrt{20 + 2\sqrt{10}}} \\
0 & \frac{3}{\sqrt{20 - 2\sqrt{10}}} & \frac{1 - \sqrt{10}}{\sqrt{20 - 2\sqrt{10}}} \\
\end{pmatrix}
\]

and \(P^T BP = P^{-1}BP = \begin{pmatrix} 6 & 0 & 0 \\
0 & 3 + \sqrt{10} & 0 \\
0 & 0 & 3 - \sqrt{10} \end{pmatrix}\).
Part 2: Definitions

Give a short but correct definition of each of the following terms by completing the lines below:

A set \( S \) is **linearly independent** iff
Answer: None of its elements can be written as a non-trivial linear combination of its other elements.

A set \( S \subset V \) **spans** \( V \) iff
Answer: Any element of \( V \) can be written as a linear combination of elements of \( S \).

A set \( S \) is a **basis** of a vector space \( V \) iff
Answer: \( S \) is linearly independent and spans \( V \).

The **dimension** of a vector space \( V \) is
Answer: The number of elements in any basis.

The **rank** of a matrix \( A \) is
Answer: The number of non-zero rows in its reduced row echelon form.

The **nullity** of a matrix \( A \) is
Answer: The dimension of its null space.

A number \( \lambda \) is an **eigenvalue** of the linear transformation \( A \) iff
Answer: There exists a vector \( v \) such that \( Av = \lambda v \).

A matrix \( P \) **diagonalizes** the matrix \( A \) iff
Answer: \( P^{-1}AP \) is a diagonal matrix.