

1. (10 pts) Find values of  $a$  and  $b$  such that the polynomial  $p(x) = ax^4 + bx$  has a horizontal tangent at the point  $(1,3)$ .

Answer:  $a = -1, b = 4$

2. (20 pts) For which values of  $a$  and  $b$  does the following linear system

$$ax_1 + x_2 + x_3 = 4$$

$$x_1 + x_2 = 6$$

$$x_1 + x_3 = b$$

- a. have one solution?      Answer:  $a \neq 2$   
 b. have no solutions?      Answer:  $a = 2, b \neq -2$   
 c. have infinitely many solutions?      Answer:  $a = 2, b = -2$

3. (10 pts) Find the following determinant

$$\begin{vmatrix} 5 & 31 & 0 & 0 & 8 \\ 2 & 8 & 0 & 0 & 7 \\ 75 & -87 & 1 & 86 & -21 \\ 0 & 1 & 0 & 0 & 0 \\ 38 & -47 & 0 & 1 & 92 \end{vmatrix}$$

Answer: 19

4. (20 pts) Find the inverses of the following matrices

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1.5 & -2 \\ -.5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 1.5 & -2 & 0 \\ -.5 & 1 & 0 \\ 0 & 0 & .2 \end{pmatrix}$$

5. (20 pts) A linear transformation  $\mathbf{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  sends  $\mathbf{e}_1$  to  $2\mathbf{e}_1 - \mathbf{e}_2$ ,  $\mathbf{e}_2$  to  $-3\mathbf{e}_1$ , and  $\mathbf{e}_3$

to  $5\mathbf{e}_3$ . What is the standard matrix of  $\mathbf{T}$ ? What is the rank of  $\mathbf{T}$ ?

Answer:  $[T] = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ . The rank of  $[T]$  is 3.

6. (30 pts) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 12 & 15 \\ 1 & 2 & 4 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 3 & 6 & 10 & 30 & 0 & 0 & 0 \\ 1 & 2 & 3 & 9 & 0 & 0 & 0 \end{pmatrix}$$

- Find the rank and nullity of  $A$  and of  $A^T$ .
- Find bases for the row space and column space of  $A$ .
- Find bases for  $N = (\text{null space of } A)$  and for  $N^\perp$ .

Answer: The rref of  $A$  is  $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ . Thus  $\text{rank}(A) = \text{rank}(A^T) = 3$ . By the

dimension theorem,  $\text{rank}(A) + \text{nullity}(A) = 7$ , thus nullity of  $A$  is 4. Similarly,  $\text{rank}(A^T) + \text{nullity}(A^T) = 7$ , and thus nullity of  $A^T$  is 4.

A basis for the  $\text{row space}(A)$  is  $\{(1, 2, 0, 0, 0, 0, 0), (0, 0, 1, 3, 0, 0, 0), (0, 0, 0, 0, 1, 4, 5)\}$ .

A basis for the  $\text{column space}(A)$  is  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \\ 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ .

A basis for the nullspace  $N$  of  $A$  is

$\{(-2, 1, 0, 0, 0, 0, 0), (0, 0, -3, 1, 0, 0, 0), (0, 0, 0, 0, -4, 1, 0), (0, 0, 0, 0, -5, 0, 1)\}$ .

Since  $N = \text{row space}(A)^\perp$ ,  $N^\perp$  is the same as  $(\text{row space}(A)^\perp)^\perp = \text{row space}(A)$ . Thus a basis for it is  $\{(1, 2, 0, 0, 0, 0, 0), (0, 0, 1, 3, 0, 0, 0), (0, 0, 0, 0, 1, 4, 5)\}$ .

7. (20 pts) (a) What is the transition matrix  $P$  from the basis

$$B' = \{\mathbf{u}'_1 = 2\mathbf{e}_1 - \mathbf{e}_2, \mathbf{u}'_2 = -3\mathbf{e}_1\}$$

to the standard basis  $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ ?

(b) If  $[u]_B = [1, 2]$  and  $[v]_{B'} = [3, 4]$ , find  $[u]_{B'}$  and  $[v]_B$ .

Answer:  $P = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}$ .  $[u]_{B'} = [-2, -5/3]$ , and  $[v]_B = [-6, -3]$ .

8. (20 pts) (a) Find an orthonormal basis for  $\mathbb{R}^2$  by applying the Gram-Schmidt orthonormalization process to the basis  $\{(3, 4), (1, 1)\}$ .

Answer:  $v_1 = (3, 4)$ , and  $u_1 = v_1/\|v_1\| = (3/5, 4/5)$ .

$v_2 = (1, 1) - \langle (3/5, 4/5), (1, 1) \rangle (3/5, 4/5) = (4/25, -3/25)$  and so  $u_2 = v_2/\|v_2\| = (4/5, -3/5)$ .

(b) Find the inner product between the vectors  $(3, 4)$  and  $(1, 1)$  for the inner product generated by  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ .

Answer:

$$\langle (3, 4), (1, 1) \rangle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) \cdot \left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = (3, 8) \cdot (1, 2) = 19.$$

9. (20 pts) Find the eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

and indicate the algebraic and geometric multiplicity for each eigenvalue you found.

$$\begin{aligned} \text{Answer: } \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 3 & \lambda - 3 \end{vmatrix} &= \lambda \begin{vmatrix} \lambda & -1 \\ 3 & \lambda - 3 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 0 \\ \lambda & -1 \end{vmatrix} = \lambda[\lambda(\lambda - 3) + 3] - 1 \\ &= \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 \end{aligned}$$

Thus the only eigenvalue is 1 and its algebraic multiplicity is 3. Its geometric multiplicity is

$$\text{nullity} \left( 1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \right) = \text{nullity} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 3 & -2 \end{pmatrix}$$

Putting this into reduced row-echelon form, this equals

$$\text{nullity} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 3 - \text{rank} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 1. \text{ Thus the geometric multiplicity is 1.}$$

10. (30 pts) Find an orthogonal matrix  $P$  that diagonalizes  $B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{pmatrix}$  and find  $P^T B P$ .

Answer: The characteristic polynomial is  $\begin{vmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda - 2 & -3 \\ 0 & -3 & \lambda - 4 \end{vmatrix} = (\lambda - 6)[(\lambda - 2)(\lambda - 4) - 9] = (\lambda - 6)(\lambda - 3 + \sqrt{10})(\lambda - 3 - \sqrt{10})$ .

To find each eigenspace, we calculate the nullspace of  $(\lambda I - B)$ . For  $\lambda = 6$ , we want the

nullspace of the matrix  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -3 & 2 \end{pmatrix}$ , which is spanned by  $(1, 0, 0)$ . For  $\lambda = 3 + \sqrt{10}$ , we

want the nullspace of the matrix  $\begin{pmatrix} -3 + \sqrt{10} & 0 & 0 \\ 0 & 1 + \sqrt{10} & -3 \\ 0 & -3 & -1 + \sqrt{10} \end{pmatrix}$ , which is spanned

by  $(0, 3, 1 + \sqrt{10})$ . A unit vector in this direction is  $(0, \frac{3}{\sqrt{20+2\sqrt{10}}}, \frac{1+\sqrt{10}}{\sqrt{20+2\sqrt{10}}})$ . Similarly, for

$\lambda = 3 - \sqrt{10}$  the easy eigenvector to write down is  $(0, 3, 1 - \sqrt{10})$  which, when normalized to unit length becomes  $(0, \frac{3}{\sqrt{20-2\sqrt{10}}}, \frac{1-\sqrt{10}}{\sqrt{20-2\sqrt{10}}})$ . Thus the orthogonal matrix that diagonalizes

$B$  is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{20+2\sqrt{10}}} & \frac{3}{\sqrt{20-2\sqrt{10}}} \\ 0 & \frac{1+\sqrt{10}}{\sqrt{20+2\sqrt{10}}} & \frac{1-\sqrt{10}}{\sqrt{20-2\sqrt{10}}} \end{pmatrix}$$

$$\text{and } P^T B P = P^{-1} B P = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 + \sqrt{10} & 0 \\ 0 & 0 & 3 - \sqrt{10} \end{pmatrix}.$$

## Part 2: Definitions

Give a short but correct definition of each of the following terms by completing the lines below:

A set  $S$  is *linearly independent* iff

Answer: None of its elements can be written as a non-trivial linear combination of its other elements.

A set  $S \subset V$  *spans*  $V$  iff

Answer: Any element of  $V$  can be written as a linear combination of elements of  $S$ .

A set  $S$  is a *basis* of a vector space  $V$  iff

Answer:  $S$  is linearly independent and spans  $V$ .

The *dimension* of a vector space  $V$  is

Answer: The number of elements in any basis.

The *rank* of a matrix  $A$  is

Answer: The number of non-zero rows in its reduced row echelon form.

The *nullity* of a matrix  $A$  is

Answer: The dimension of its null space.

A number  $\lambda$  is an *eigenvalue* of the linear transformation  $A$  iff

Answer: There exists a vector  $v$  such that  $Av = \lambda v$ .

A matrix  $P$  *diagonalizes* the matrix  $A$  iff

Answer:  $P^{-1}AP$  is a diagonal matrix.