Math 254 Practice Final

1. (10 pts) Find values of $a$ and $b$ such that the polynomial $p(x) = ax^4 + bx$ has a horizontal tangent at the point (1,3).

2. (20 pts) For which values of $a$ and $b$ does the following linear system

$$ax_1 + x_2 + x_3 = 4$$
$$x_1 + x_2 = 6$$
$$x_1 + x_3 = b$$

a. have one solution?
b. have no solutions?
c. have infinitely many solutions?

3. (10 pts) Find the following determinant

$$\begin{vmatrix}
5 & 31 & 0 & 0 & 8 \\
2 & 8 & 0 & 0 & 7 \\
75 & -87 & 1 & 86 & -21 \\
0 & 1 & 0 & 0 & 0 \\
38 & -47 & 0 & 1 & 92
\end{vmatrix}$$

4. (20 pts) Find the inverses of the following matrices

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}^{-1}$$

5. (20 pts) A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ sends $e_1$ to $2e_1 - e_2$, $e_2$ to $-3e_1$, and $e_3$ to $5e_3$. What is the standard matrix of $T$? What is the rank of $T$?
6. (30 pts) Given the matrix

\[
A = \begin{pmatrix}
1 & 2 & 0 & 0 & 3 & 12 & 15 \\
1 & 2 & 4 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 5 \\
3 & 6 & 10 & 30 & 0 & 0 & 0 \\
1 & 2 & 3 & 9 & 0 & 0 & 0
\end{pmatrix}
\]

a. Find the rank and nullity of \( A \) and of \( A^T \).
b. Find bases for the row space and column space of \( A \).
c. Find bases for \( N=(\text{null space of } A) \) and for \( N^\perp \).

7. (20 pts) (a) What is the transition matrix \( P \) from the basis

\[
B' = \{ u'_1 = 2e_1 - e_2, u'_2 = -3e_1 \}
\]
to the standard basis \( B = \{ e_1, e_2 \} \)?

(b) If \( [u]_B = [1, 2] \) and \( [v]_{B'} = [3, 4] \), find \( [u]_{B'} \) and \( [v]_B \).

8. (20 pts) (a) Find an orthonormal basis for \( \mathbb{R}^2 \) by applying the Gram-Schmidt orthonormalization process to the basis \{\((3, 4), (1, 1)\)\}.

(b) Find the inner product between the vectors \((3, 4)\) and \((1, 1)\) for the inner product generated by \( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \).

9. (20 pts) Find the eigenvalues of the matrix

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -3 & 3
\end{pmatrix}
\]

and indicate the algebraic and geometric multiplicity for each eigenvalue you found.

10. (30 pts) Find an orthogonal matrix \( P \) that diagonalizes \( B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{pmatrix} \) and find \( P^T BP \).
Part 2: Definitions

Give a short but correct definition of each of the following terms by completing the lines below:

A set $S$ is *linearly independent* iff

A set $S \subset V$ *spans* $V$ iff

A set $S$ is a *basis* of a vector space $V$ iff

The *dimension* of a vector space $V$ is

The *rank* of a matrix $A$ is

The *nullity* of a matrix $A$ is

A number $\lambda$ is an *eigenvalue* of the linear transformation $A$ iff

A matrix $P$ *diagonalizes* the matrix $A$ iff