Digital Systems & Transforms

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Huang: 5-5.2.1, 5.2.4, 5.3.1, 5.3.5, 5.4.2, 5.5-5.5.2

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Digital Systems

When

- x[n] is an input signal varying with discrete time index n, and
- T is a system that transforms the input x[n] to an output y[n],

we call this a *digital system* and write:

$y[n] = T\{x[n]\}$

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Filtering

- Transforms are called filters
- Some useful filters
  – low pass
  – band pass
  – high pass
  – homomorphic [later in the semester]

Linear Time Invariant (LTI) Systems

- A system is *linear* iff
  $T\{ax_1[n]+bx_2[n]\} = aT\{x_1[n]\}+bT\{x_2[n]\}$
  (the superposition & linear scaling properties)

- A system is *time-invariant* iff
  $y[n-n_0] = T\{x[n-n_0]\}$

- We will be primarily concerned with a type of transform called a finite impulse response filter.

Finite Impulse Response (FIR) Filters

- FIR filter

h[0]=1 h[1]= -.95

x[2] x[1] x[0]

x[n] = 0 outside 0,1,2

1*0 -.95*0

Initial conditions

y[-1]=1*0+-.95*0

x[2] x[1]

1*x[0] -.95*0

x[0] enters system

y[0]=1*x[0]+-.95*0

x[2]

1*x[1] -.95*x[0]

y[1]=1*x[1]+-.95*x[0]

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FIR Filters

1*x[2] -.95*x[1]

y[2]=1*x[2]+-.95*x[1]

1*0 -.95*x[2]

y[3]=1*0+-.95*x[2]

1*0 -.95*0

y[4]=1*0+-.95*0

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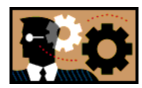
Convolution

- The operations of the previous slides can be written as:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

where h[n-k] is the $(n\text{-}k)^{th}$ filter tap.



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Convolution

- An input signal is *convolved* with filter {h[n]} to produce output: y[n]=x[n]*h[n]

- Convolution is commutative, associative, and distributive

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Convolution in Matlab

```
>> % Input
>> x = [17, 22, 25, 12, -8, 10];
>> % Filter impulse response (tap weights)
>> h = [1, -.95];
>> conv(x, h)        % x * h – convolve x with h
ans =
   17.0000    5.8500    4.1000  -11.7500  -19.4000
    17.6000   -9.5000
```


Complex exponentials

- One signal of special interest is the complex exponential

$$
\begin{aligned}
x[n] &= Ae^{j\omega n} \\
&= A\cos(\omega n) + jA\sin(\omega n)
\end{aligned}
$$

where ω represents the frequency in radians per second and *j* denotes imaginary

[Matlab phasor demo]

This presentation follows that of fred harris's "Trigonometric Transformations," TR DSP-005, 8/1981, Scientific-Atlanta.

Exponentials & sinusoids

- Suppose we add two exponentials which differ only by rotational direction:

$$
\begin{aligned}
x[n] &= Ae^{j\omega n} + Ae^{-j\omega n} \\
&= A\cos(\omega n) + jA\sin(\omega n) + A\cos(\omega n) + jA\sin(-\omega n) \\
&= 2A\cos(\omega n) + jA\sin(\omega n) - jA\sin(\omega n) \\
&= 2A\cos(\omega n)
\end{aligned}
$$

$$A\cos(\omega n) = \frac{Ae^{j\omega n} + Ae^{-j\omega n}}{2}$$

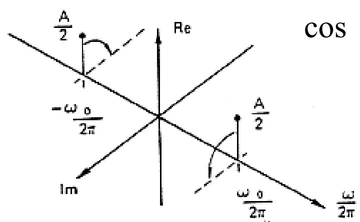
Exponentials and sinusoids

- Similarly, subtraction of the same two exponentials can be shown to be related to sine waves:

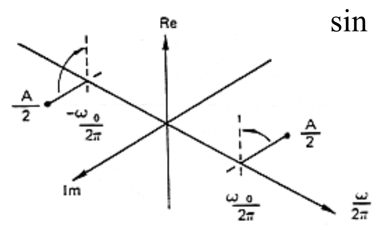
$$A\sin(\omega n) = \frac{Ae^{j\omega n} - Ae^{-j\omega n}}{2j}$$

harris's scoreboard representation

- Cosines/sines are linear combinations of exponentials of half the magnitude



cos



sin

these differ only by phase

harris TR DSP-005

Convolving a filter with an exponential

- Let $x[n] = e^{j\omega_0 n}$

- Then:

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= h[n] * x[n] && \text{commutative} \\
&= \sum_{k=-\infty}^{\infty} h[k] x[n-k] && \text{defn } *
\end{aligned}
$$

Convolving a filter with an exponential

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \qquad \text{defn } x[n-k] = e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \qquad j\omega_0 \text{n is constant}$$

$$= e^{j\omega_0 n} H(e^{j\omega_0}) \qquad \text{where } H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

- The output y[n] is another exponential of the same frequency scaled by $H(e^{j\omega_0})$

Eigenvalues & Eigensignals

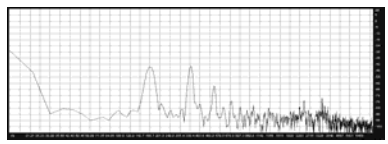
- Complex exponentials are *eigensignals* of LTI systems. An eigensignal convolved with an LTI system produces another eigensignal.
- $H(e^{j\omega_0})$, a complex quantity, is an *eigenvalue* of an LTI system. It indicates the quantity by which the eigensignal is scaled.

Fourier Transform

- The continuous frequency Fourier Transform of x[n] is:

$$X(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega_0 k}$$

- This enables us to evaluate the frequency content of x, at *any* frequency ω_0.



dB

log Hz

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Notes of the Fourier Transform

- The Fourier transform for aperiodic signals exists when: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

- It is sometimes convenient to normalize the frequency range to [-π, π]

- To move from a normalized [-π, π] frequency axis to Hz:

$$f(\omega) = \frac{\omega}{2\pi} \times SamplingRate, \;\; \omega \in [0, 2\pi]$$

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Notes on the Fourier transform

- Output of Fourier transform is a set of complex sinusoids, which are eigensignals for LTI systems.
- Recall that LTI systems only *scale* eigensignals, so an LTI system can *only* boost or attenuate existing frequencies in a signal, not add new ones.

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Notes on the Fourier Transform The Convolution Property

- Time domain convolution ↔
  Frequency domain multiplication
  $x[n]*h[n] = X(\omega)H(\omega)$
- Time domain multiplication ↔
  Frequency domain convolution
  $x[n]h[n] = X(\omega)*H(\omega)$

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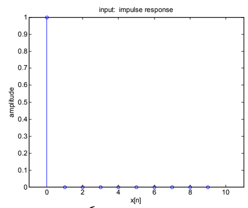
Unit impulse & impulse response

- Consider unit impulse (aka Kronecker delta)

$$\delta[n] * h[n] = h[n]$$



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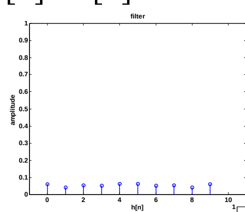
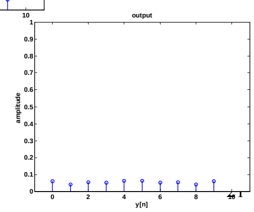


$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

=



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Significance of the impulse response

- Completely characterizes an LTI system
- The Fourier transform of the impulse response lets us understand what the system does.

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Discrete Frequency & Short-Time Fourier Transforms

- Computational methods require a discrete frequency domain.
- Speech signals change over time and we want to analyze a "static" portion of the signal.

DFT - Discrete Fourier Transform

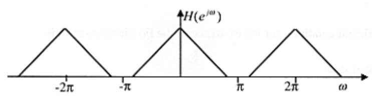
- Assume $x_N[n] = x_N[n+N]$
  - $x_N[n]$ periodic with period N.
- We define the DFT as

$$X_N[k] = \sum_{n=0}^{N-1} x_N[n] e^{-j2\pi nk/N} \text{ where } 0 \le k < N$$

and will refer to this as Fourier analysis

Notes on the DFT

- The DFT is periodic with period 2π.
- $X_n[k]$ samples the frequency domain.



Huang et al. p 209

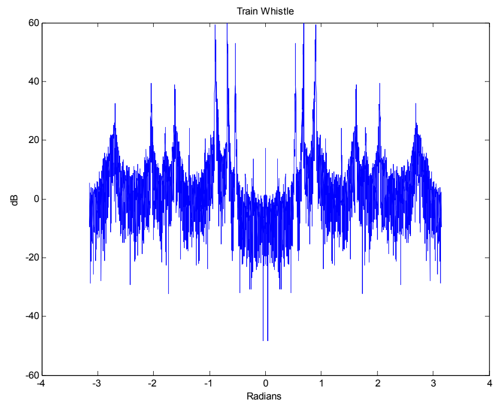
- An alternate form exists for analyzing periodic signals.

Generating a DFT Plot in Matlab

```
% train whistle into y, Fs
% y=signal, Fs=Sample Rate
% train is a standard demo
load train;
sound(y, Fs)   % play it!

% discrete Fourier transform
% (even though name is fft)
Y = fft(y);    % 0 -> 2pi
Y = fftshift(Y); % -pi->pi

% Take magnitude^2
YMagSq = Y .* conj(Y);

% Convert to dB
YdB = 10 * log10(YMagSq);
```

```
% number of frequency bins
N = length(Y);

% Create a normalized freq
% range [-pi, pi)

NormFreq = -pi : 1/N * 2*pi : pi -
   1/N;

% Create a new figure &
% name the window
figure('Name', 'Choo-choo');

plot(NormFreq, YdB);
xlabel('Radians')
ylabel('dB')
title('Train Whistle')
```

DFT Plot



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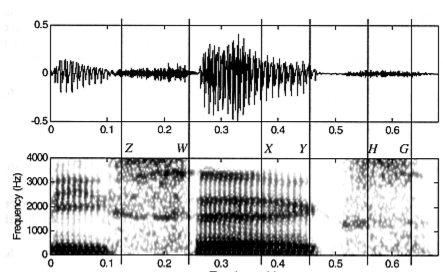
Short Time Fourier Analysis

- Analyzing the whole signal doesn't make sense, not all regions are similar:



Huang et al. p 277

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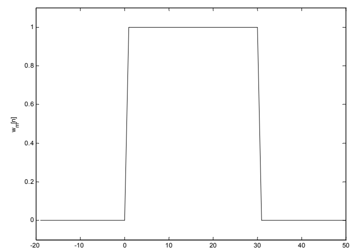
Short-Time Fourier Analysis

- Segment the signal into small *frames* and then perform an analysis on each frame.
- Let $w_m[n]$ be a rectangular window for the m'th frame.



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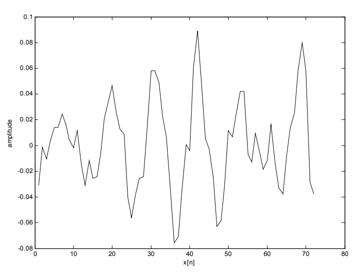
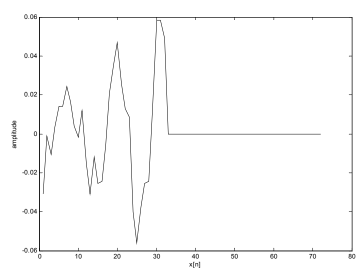
Extracting the short-time signal

- x[n]
- $x_m[n]=x[n]w_m[n]$





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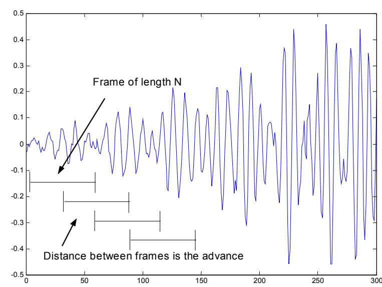
Short-time Fourier Analysis

- By shifting the region where the window is non-zero we can control what portion of the signal remains for each frame.
- We must choose:
  - Frame length
  - Frame advance



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Short-Time Fourier Analysis

- Fourier analysis of the framed signal:

$$X_m(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]w_m[n]e^{-j\omega n}$$

- Analysis with discrete frequencies:

$$X_m[k] = \sum_{n=0}^{N-1} x[n]w_m[n]e^{-j2\pi nk/N} \text{ where } 0 \le k < N$$

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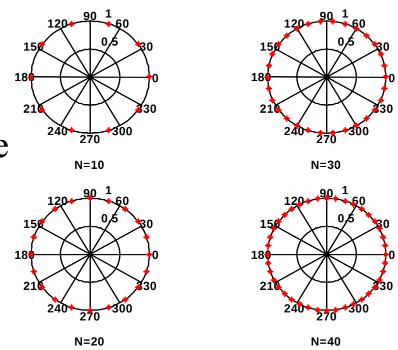
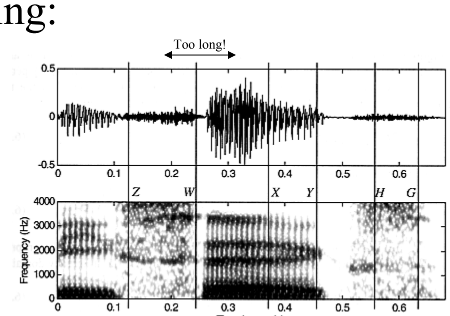
The time frequency trade-off

- As the window length increases, the number of frequencies at which we approximate increases:



Time-Frequency Trade-Off

- but a longer window increases the chance of smearing:



Huang et al. p 277

Window Considerations

- The approximation does not work very well if we do not have at least one pitch period in the window.
- Accommodating lower pitched voices means longer windows.

Window Considerations

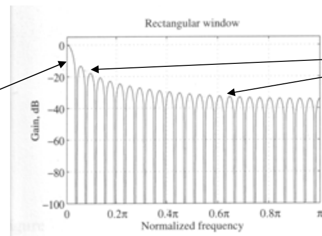
- Recall: Multiplication in the time domain is convolution in the frequency domain

$$x[n]w[n] \leftrightarrow X(\omega)*W(\omega)$$

- Ideally, we would like a window whose transforms was narrow with no side band information.

Spectral leakage

- Consider W(ω) when w[n] is a rectangular window:



Narrow main lobe

side lobes relatively high

Mitra 2001, p. 453

- During the convolution, the side lobes will result in the DFT having additional energy to either side of a peak. This is called spectral leakage.

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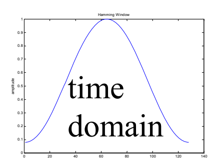
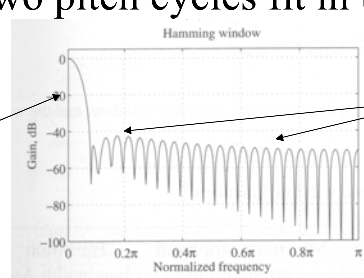
Spectral leakage



time domain

- Hamming window has smaller side lobes at the expense of a wider main lobe.
- Requires two pitch cycles fit in the window.



Wider main lobe

side lobes much lower

Mitra 2001, p. 453

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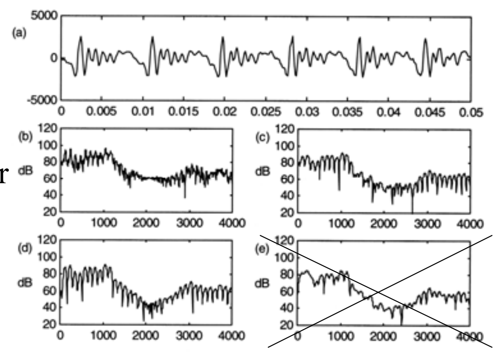
Spectral leakage comparison

Rectangular

Hamming



Time domain
F0 110 Hz
(9 ms period)

too short!
2 cycles = 18 ms

30 ms 15 ms

window length

Huang et al. 2001, p. 279

Typical framing parameters

- Window: Hamming.
- Frame/Window length: 20-30 ms.
- Frame advance: 10-20 ms.

Inverse DFT

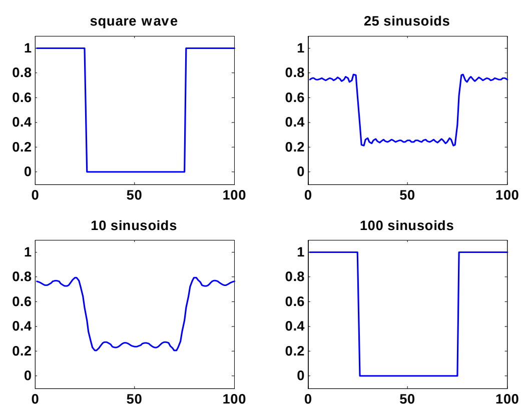
- Given a set of complex exponential coefficients X[n], we can recover the signal x[n]:

$$x_N[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_N[k]e^{j2\pi k/N} \text{ where } 0 \le n < N$$

- Signal recovery is called Fourier synthesis.

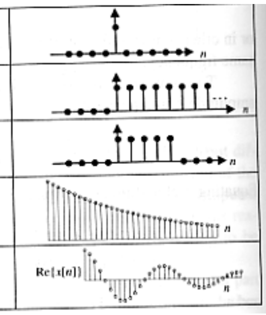
Approximate Synthesis Retaining <N Fourier Coefficients

- Periodic square wave N=100
- Signal synthesis using varying numbers of sinusoids



A few useful signals

Kronecker delta, or unit impulse	$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$	
Unit step	$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$	
Rectangular signal	$\text{rect}_N[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & otherwise \end{cases}$	
Real exponential	$x[n] = a^n u[n]$	
Complex exponential	$x[n] = a^n u[n] = r^n e^{jn\omega_0} u[n]$ $= r^n (\cos n\omega_0 + j \sin n\omega_0) u[n]$	



Huang et al. p. 206

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Convolution & Periodic Signals

- When a signal is periodic, the convolution operator

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

is undefined as the sum would be infinite.

- For periodic signals, *circular convolution* is used which sums over one period only:

$$x[n] \otimes h[n] = \sum_{k=<N>} x[k] h[n-k]$$

where <N> is any period of x (i.e. 0:N-1)

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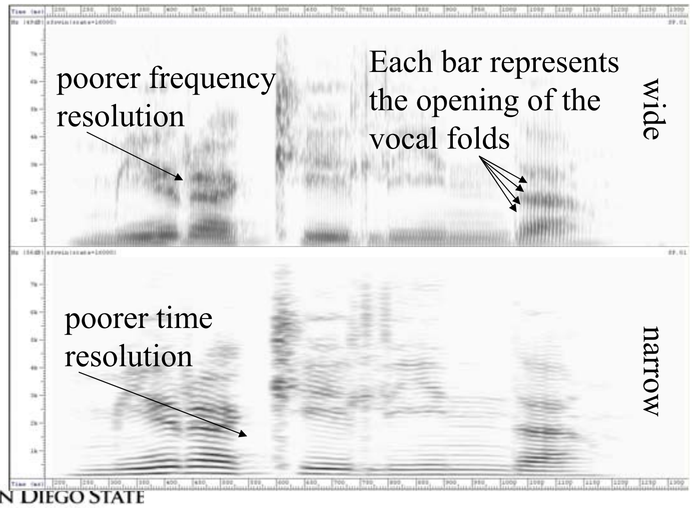
Spectrograms

- Spectrograms are a series of short-time DFTs on windowed segments of a signal
- There are two types of spectrograms:
  - Narrow-band spectrogram
  - Wide-band spectrogram
- The type depends upon the window length.

Window length

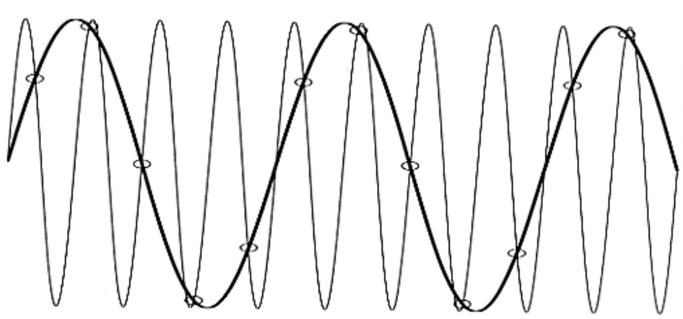
- Short ≈ 10 ms or less
  - Better time analysis
  - Frequency bands are widely spaced.
  - F0 can be estimated by determining time between vertical striations
- Long ≈ 20 ms or more
  - Poor time analysis
  - Frequency bands narrowly spaced.

Narrow vs. Wide-band Spectrogram



Sampling

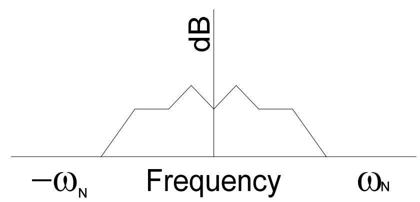
- More than one way to interpolate a set of samples:



Karu p. 98

Nyquist Sampling Theorem

- If an analog signal x(t) is bandlimited between $\pm\omega_N$:



- Then x(t) can be accurately determined by samples x[n] whenever the sampling rate ω_S is > 2 ω_N.

Nyquist Sampling Theorem

- ω_N is called the Nyquist rate.
- Analog signals frequently contain frequencies higher than the rate at which we want to sample.
- Consequently, frequencies higher than ω_N must be eliminated before sampling. This is done through low-pass filtering.

Parseval's Theorem

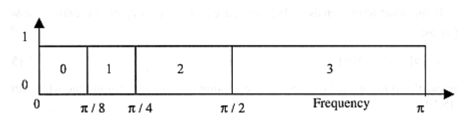
- The total energy in a signal can be useful

- Parseval's Theorem states that energy can be calculated in either the time or frequency domain and that the two are equivalent:

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \left| X(\omega) \right|^2 \delta\omega$$

Filterbanks

- A filterbank is a set of filters than span the spectrum.
- The filterbank decomposes the signal into multiple signals, each of which covers some subset of the total bandwidth:



Huang p. 255

Filterbanks

- Constructing a filterbank is beyond the scope of this class, but we will use the idea later in the semester when we speak about Mel-filtered cepstral coefficients.
- A filterbank can be constructed from a DFT.

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Discrete Cosine Transform (DCT)

- The DCT is a transform similar to the Fourier transform.
- The DCT decomposes a signal into multiples of cosines.

$$C[k] = \sum_{n=0}^{N-1} x[n]\cos(\pi k(n+1/2)/N) \quad \text{where } 0 \le k < N$$

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DCT vs. DFT

- The DCT represents more of the signal with fewer coefficients than the DFT (called energy compaction).
- We will use the DCT later in the semester when we compute a feature known as the *cepstrum*. (Matlab: function dct)