1 Hermitian Matrices

We can write any complex matrix as the sum of it’s real part and imaginary part $A = \text{Re}(A) + i\text{Im}(A)$, where $\text{Re}(A), \text{Im}(A)$ are both in $\mathcal{M}_n(\mathbb{R})$. We will see that there is a similar decomposition based on the eigenvalues of $A$.

Suppose that $\lambda$ is an eigenvalue of $A$. Then $\lambda$ is a root of the minimal polynomial of $A$ (and conversely, if $\lambda$ is a root of the minimal polynomial then $\lambda$ is an eigenvalue of $A$). The minimal polynomial of $A$ and of $A^T$ are the same, so $\lambda$ must also be an eigenvalue of $A^T$.

Taking conjugate transposes we can see that $\bar{\lambda}$ is an eigenvalue of $A^*$. 

$$Au = \lambda u \iff u^* A^* = \bar{\lambda} u^*$$

**Definition 1.1.** A matrix $H \in \mathcal{M}_n$ is Hermitian when $H^* = H$, that is $\overline{h_{j,i}} = h_{i,j}$. A matrix is skew-Hermitian when $H^* = -H$. Over the reals, the conjugate is unnecessary and the terminology is different. A matrix $A \in \mathcal{M}_n(\mathbb{R})$ is symmetric when $A^T = A$ and skew symmetric when $A^T = -A$.

**Observations:** There are a number of quick computations that reveal interesting properties of Hermitian matrices.

1. $A$ is Hermitian if and only if $iA$ is skew Hermitian.

2. The diagonal entries of a Hermitian matrix must be real. The diagonal entries of a skew-Hermitian matrix must be purely imaginary.

3. The set of Hermitian matrices is closed under addition and scalar multiplication by a real number. So, the set of Hermitian matrices is a vector space over $\mathbb{R}$. 
(4) The set of Hermitian matrices is not closed under multiplication. (Find
A, B, both Hermitian, such that AB is not.)

(5) For any matrix A, A + A* and AA* are Hermitian, and A − A* is
skew-Hermitian.

(6) We may write A as the sum of a Hermitian matrix and a skew-
Hermitian matrix

\[ A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) \]

The final item was alluded to at the beginning of this section. It gives a
decomposition of a matrix A into a Hermitian part and a skew-Hermitian
part. The following proposition justifies thinking of this decomposition of
A as a “real” part (the eigenvalues are real) and an “imaginary” part (the
eigenvalues are pure imaginary).

**Proposition 1.2.** The following properties hold for a Hermitian matrix A.

1. For all \( v \in \mathbb{C}^n \), \( v^*Av \in \mathbb{R} \).

2. The eigenvalues of A are real.

3. Let \( u, w \) be eigenvectors for distinct eigenvalues \( \lambda, \mu \), respectively. Then
   \( v^*u = 0 \).

4. For any matrix S, \( S^*AS \) is Hermitian.

**Proof.** For any vector \( v \in \mathbb{C}^n \),

\[ (v^*Av)^* = v^*A^*v = v^*Av \]

so \( v^*Av \) must be real.

Let \( \lambda, u \) be an eivenvalue and associated eigenvector.

\[ u^*Au = u^*\lambda u = \lambda u^*u \]

By part (1) this must be real. Since \( U^*u \) is also real, \( \lambda \) must be real.

Let \( u, w \) be eigenvectors for distinct eigenvalues \( \lambda, \mu \), which by (2) must
be real. Then \( u^*Aw = u^*\mu w \), since but also \( u^*Aw = \lambda u^*w \). Thus \( \lambda u^*w = 
\mu u^*w \). Since \( \lambda \neq \mu \), we must have \( u^*w = 0 \).

For any S, \( (S^*AS)^* = S^*A^*S = S^*AS \), so \( S^*AS \) is Hermitian.
Exercises 1.3.

(a) Show that the eigenvalues of a skew-Hermitian matrix are pure imaginary.

(b) If $A$ is Hermitian, then so is $A^k$ for $k \geq 0$. If $A$ Hermitian and also invertible $A^k$ is Hermitian for all $k \in \mathbb{Z}$.

(c) Let $A, B$ be Hermitian. Show that $A$ and $B$ are similar iff they are unitarily similar. [H4.1ex2].

(d) Let $A, B$ be Hermitian.

(1) Show that $AB - BA$ is skew-Hermitian.

(2) Show that $\text{Tr}((AB)^2) = \text{Tr}(A^2B^2)$. [H4.1ex11] Hint: Consider $\text{Tr}((AB - BA)^2)$.