Gersgorin Theorem

March 19, 2013
We begin by establishing some notation while giving a theorem from math 254 (without proof).

Theorem 1: For \( A = [a_{ij}] \in M_n(F) \), the following are equivalent:

1. \( A \) is nonsingular (i.e., \( Ax = 0 \) only when \( x = 0 \))
2. \( \text{nullity} (A) = 0 \)
3. \( \text{rank} (A) = n \)
4. \( A \) is invertible (i.e., there exists \( B \in M_n(F) \) with \( AB = BA = I_n \))
4. \( \det (A) \neq 0 \)
5. \( \text{RREF}(A) = I_n \)
6. 0 is not an eigenvalue of \( A \)

Remarks:

i) Theorem 1 holds for any field \( F \), but we are interested when \( F = \mathbb{C} \) (or \( \mathbb{R} \)).

ii) In what follows, conditions 1 and 6 of Theorem 1 will be most pertinent.
$A \in M_n(\mathbb{C})$ is diagonally dominant in row $i$ if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$ 

$A$ is simply diagonally dominant if $A$ is diagonally dominant in row $i$ for all $i = 1, 2, \ldots, n$.

Remarks

The above definition may also be given as:

- $A$ is diagonally dominant of rows
- $A$ is strongly diagonally dominant

We say $A$ is weakly diagonally dominant (of rows) if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad \text{all } i = 1, 2, \ldots, n.$$
Theorem 2 (Fergy-Desplanques) If \( A \in M_n(\mathbb{C}) \) is diagonally dominant, then \( A \) is nonsingular.

**Proof**

We will show the contrapositive that if \( A \) is singular, then \( A \) is not diagonally dominant.

Suppose \( A \) is singular and \( \circ \neq x \in \mathbb{C}^n \) satisfies \( Ax = 0 \). We may divide \( x \) by one of its largest entries so we may assume that

\[
1 = \max \{ |x_1|, |x_2|, \ldots, |x_n| \} = |x_i|.
\]

Since \( Ax = 0 \), \( (Ax)_i = 0 \) or

\[
a_{ii} x_i + a_{i2} x_2 + \cdots + a_{in} x_n = 0
\]

Solving this equation for \( a_{ii} \) yields

\[
a_{ii} = -\sum_{j \neq i} a_{ij} x_j
\]

so that

\[
|a_{ii}| \leq \sum_{j \neq i} |a_{ij}| |x_j| \leq \sum_{j \neq i} |a_{ij}|
\]

which shows that \( A \) is not diagonally dominant in row \( i \).
We will use theorem 2 to prove Zerosgerin's Theorem which comes next.

For any $z_0 \in \mathbb{C}$ and $r > 0$, let

$$D(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| \leq r \}$$

be the closed disc of radius $r$ with center $z_0$.

For any $A \in M_n(\mathbb{C})$, let

$$r_i = \frac{\sum_{j \neq i} |a_{ij}|}{n}$$

and let

$$D_i = D(a_{ii}, r_i)$$

be the Zerosgerin discs of $A$,

$i = 1, 2, \ldots, n$. 
Theorem 2 (Gersgorin) Every eigenvalue of $A$ lies in $D_1 U D_2 U \ldots U D_n$.

Proof:
Suppose $\lambda \in \mathbb{C}$ and $\lambda \notin D_i$, $i = 1, 2, \ldots, n$. Recall that $\lambda$ is an eigenvalue of $A$ iff $\lambda I_n - A$ is singular. We claim $\lambda$ is not an eigenvalue of $A$, or that $\lambda I_n - A$ is nonsingular.

Since $\lambda \notin D_i$, $i = 1, 2, \ldots, n$,

$$|\lambda - a_{ii}| > r_i = \sum_{j \neq i} |a_{ij}|$$

for all $i = 1, 2, \ldots, n$. This shows that $\lambda I_n - A$ is diagonally dominant and hence nonsingular.

Here we note that the Gersgorin Region $D_1 U D_2 U \ldots U D_n$ consists exactly of those $\lambda$ for which $\lambda I_n - A$ is not diagonally dominant.

Remark: $A$ and $\lambda I_n - A$ have the same $r_i$. 
The last theorem needs to have some ideas from analysis and "Hersprung's Theorem" frequently refers to theorems 3 and 4 together.

**Lemma 1** Suppose $A(t)$, $0 \leq t \leq 1$ is a continuous function from $[0, 1]$ to $M_n(C)$. Then there exists continuous eigenfunctions,

$$\lambda_i : [0, 1] \rightarrow C$$

such that the eigenvalues of $A(t)$ are $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$.

**Lemma 1** more or less follows from the following facts:

1. The coefficients of the characteristic polynomial are continuous functions of the entries of the matrix.
2. The roots of a polynomial are continuous functions of its coefficients.
The Yersgorin Region $D_1D_2\ldots D_n$ may be comprised of anywhere from 1 to $n$ connected components.

For example, if $A$ is 8-by-8 with Yersgorin Discs

Then there would be four such components. One of these is a single disc; two of these are unions of two-discs; and one of these is a union of three discs.

**Theorem 4:** Let $C$ be a connected component of $D_1D_2\ldots D_n$ which is a union of $k$ of the discs. Then $C$ contains exactly $k$ eigenvalues of $A$, counted according to algebraic multiplicity.
proof sketch

Split \( A = D + B \) where

1. \( D \) is diagonal and \( B \) has zero diagonal

\[
\text{example: If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \text{ then }
\begin{align*}
D &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Define

\[
A(t) = D + tB, \quad 0 \leq t \leq 1
\]

so that \( A(0) = D, A(1) = B \), and \( A(t) \) is continuous (even linear!).

Let

\[
\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)
\]

be the corresponding eigenfunctions. For each \( 0 \leq t \leq 1 \), let \( G(t) \) be the Herschel Region of \( A(t) \), so that \( G(t) \) contains the eigenvalues of \( A(t) \).

For \( t = 0 \), \( A(0) = D \), so the Herschel discs all have radius 0 and the eigenvalues of \( A(0) \) are just \( \lambda_1, \lambda_2, \ldots, \lambda_n \).
In fact, we may assume that
\[ \lambda_i(t) = a_{i,k} \quad \text{all } i = 1, 2, \ldots, n. \]

Assume that the connected component \( C \) is a union of the first \( k \) Neumann discs of \( A \) : \( G = D_1 \cup D_2 \cup \cdots \cup D_k. \)

For each \( 0 \leq t \leq 1 \), the Neumann discs of \( A(t) \) have center \( a_{i,k} \) and radius \( \pm r_i \), where \( r_i = \sum_{j \neq i} |a_{i,j}| \)
so that
\[ G(0) \subseteq G(t) \subseteq G(1) = G \]
whenever \( 0 \leq d \leq t \leq 1 \).

Now, since \( \lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t) \) are continuous, none of these functions can "jump" from one component of \( G \) to another. Thus the first \( k \) eigenfunctions remain in \( C \), and none of the other \( n - k \) eigenfunctions can "jump" into \( C \).
exercise Use Gersgorin's Theorem to prove theorem 2 directly.

exercise Let \( A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \)

a) Find the Gersgorin region of \( A \), \( G \), and display \( G \) in the complex plane.

b) Find and display the eigenvalues of \( A \) on the same page as for part a.

exercise Repeat the previous exercise for \( A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \)

exercise For \( A \in M_n(\mathbb{C}) \) let
\[
\rho = \max \sum_{j=1}^{n} |a_{ij}| \]
be the maximum absolute row-sum of \( A \).
If \( \lambda \) is an eigenvalue of \( A \), show \( |\lambda| \leq \rho \).