Abstract Algebra
Math 521A
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Review for Third Exam: Chapters 5, 6

Commutative rings with identity: ideals, and congruence modulo an ideal

- Know how to
  - Define ideal, and prime ideal.
  - Prove that a subset of a ring is an ideal (or show that it isn’t).
  - Use the language of ideals with \( \mathbb{Z}, \mathbb{Z}_n, F[x], F[x]/m(x) \).

- Know the relationship between homomorphisms and ideals. (The kernel is an ideal; The first isomorphism theorem; See also 6.2 #13, 24).

- Know how to compute in the quotient of a ring by an ideal.
  - If \( I \) is an ideal in \( R \), the elements of \( R/I \) are written \( a + I \) where \( a \in R \).
  - \( a + I = b + I \) iff \( a - b \in I \).
  - Addition in \( R/I \) is defined by \((a + I) + (b + I) = (a + b) + I\).
  - Multiplication in \( R/I \) is defined by \((a + I)(b + I) = (ab) + I\).

- Know how to compute in a polynomial ring modulo a polynomial.
  - Find the inverse of an element \( a(x) \) of \( F[x]/m(x) \), when \( a(x) \) is coprime to \( p(x) \).
  - Identify units and zero divisors in \( F[x]/m(x) \).
  - Identify all ideals in \( F[x]/m(x) \).
  - Define irreducible and prime for polynomials.
  - Know that \( m(x) \) is prime iff \( m(x) \) is irreducible. In this case, \( F[x]/m(x) \) is a field.

- Be able to work in \( R \times S \) where \( R, S \) are commutative rings with identity.
  What are the ideals in this ring?

- Our standard examples of non-principal ideal rings.
  - Be able to compute and work with ideals in \( \mathbb{Z}[x] \) and \( F[x, y] \) for \( F \) a field. See 6.1#41 and example p. 140, 6.2#13, 6.3 #10.
  - Be able to find units, zero divisors, idempotent elements (\( x \) such that \( x = x^2 \)) in \( F[x, y] \) modulo a simple ideal like \( \langle x^2, xy, y^2 \rangle \).