# Linear Algebra
# Math 254

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Topics for Final Exam
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Solving Linear Systems

- Transform a system of linear equations into a matrix equation.
- Solve a system using Gaussian elimination.
- Explain the steps that you use (switching rows, scaling a row, adding a row to another one).
- Identify the matrix operation corresponding to each step.
- Find the RREF form of a matrix.
- Find the kernel of a linear transformation.
- Find the image of a linear transformation.
- Write a vector $x$ as a sum of vectors $v_1, \ldots, v_m$.
- Invert a matrix using Gaussian elimination.

Vector Space Terminology

- Definitions you should know:
  - linear combination, span, linear independent, basis;
  - subspace, linear transformation;
  - rank, nullspace (a.k.a. kernel), column space( aka image) of a matrix;
  - orthogonal, orthonormal basis,
  - eigenvalue, eigenvector, eigenspace.
- Decide whether a function is a linear transformation.
- Be able to state and use the Rank-Nullity Theorem.

Orthogonality

- In $\mathbb{R}^2$, given a unit vector $u$, know how to compute
  - the matrix for projection on to the line through $u$,
  - the matrix for reflection about the line through $u$,
- In $\mathbb{R}^2$, know how to identify a shear matrix, a scaling matrix, and a rotation matrix.
- Find the orthogonal and perpendicular components of a vector relative to a subspace. Given $V \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, decompose $x$, relative to $V$ as $x = x^\perp + x^\parallel$.
- Find the components of $x \in V$ relative to an orthonormal basis of $V$.
- Transform a given basis to an orthonormal basis using Gram-Schmidt.
- Write the $QR$ decomposition of a matrix.

Determinants

- Know the basic properties of determinants (6.1.6, 6.2.1, 6.2.2, 6.2.4, 6.2.5).
- Similar matrices have the same determinant.
- Compute an arbitrary $2 \times 2$ or $3 \times 3$ determinant.
- Compute the determinant of larger matrices with special conditions (e.g. lots of zeros).

Eigenvalues and Eigenvectors

- Find the characteristic polynomial of an $n \times n$ matrix $A$.
- Find the eigenvalues for $2 \times 2$, and (doable) $3 \times 3$ matrices and triangular matrices.
- Find the algebraic multiplicity and the geometric multiplicity of each eigenvalue.
- Find a basis for the eigenspace associated to each eigenvalue.
- Diagonalize a matrix $A$ when it is possible.
- Understand that diagonalization is change of basis.

1. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

   (a) Write $y = \begin{bmatrix} 2 \\ -3 \\ -10 \end{bmatrix}$. as a linear combination of the $v_i$. Use Gaussian elimination and identify each step as a matrix multiplication.
   (b) Find the kernel and image of the transformation $T(x) = Ax$.
   (c) What are the nullity and the rank of $A$.
   (d) Find the inverse of $A$.
   (e) Solve part (a) using $A^{-1}$.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 3 & 1 \\ 4 & 8 & 5 & 3 \end{bmatrix}$$

   (1) Find a basis for the kernel of find a basis for the image of the transformation $T(x) = Ax$.
   (b) What are the nullity and the rank of $A$?

3. Classify the geometrical properties of the following matrices (look at what each does to the standard basis).

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Find the $QR$ factorization of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

4. Compute the determinant.

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

5. Diagonalize each matrix if possible

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$