

NAME:

Abstract Algebra
Math 521A
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Review of the course

Rings and Ideals

- Know the definitions:
 - ring, commutative, identity, field;
 - unit, zero divisor, characteristic;
 - homomorphism, isomorphism.
- Know how to:
 - Prove that a subset of a ring is a subring, or an ideal (or show that it isn't).
 - Prove that a function is a homomorphism, or isomorphism (or show it isn't).
 - Show that two rings can't be isomorphic, because they have some different structure.
 - Identify the units and zero divisors in a ring.
- Know how to construct new rings from old and to compute in these rings.
 - The Cartesian product of rings R and S is a ring $R \times S$.
 - The 2×2 matrices over a ring R form a ring, which we write $M(R)$.
 - We also have the polynomial ring, $R[x]$ over a ring R .
- Know how to work with quotient rings. (Here you may assume the ring is commutative with identity.)
 - If I is an ideal in R , the elements of R/I are written $a + I$ where $a \in R$.
 - $a + I = b + I$ when $a - b \in I$.
 - Addition in R/I is defined by $(a + I) + (b + I) = (a + b) + I$.
 - Multiplication in R/I is defined by $(a + I)(b + I) = (ab) + I$.
 - I is prime if and only if R/I is an integral domain.
- Know the special properties of \mathbb{Z} and $F[x]$.
 - Division theorem.
 - Euclidean algorithm.
 - Prime iff irreducible.
 - Unique factorization.
 - Every nonzero element is either a zero divisor or a unit.
 - In $F[x]$, $(x - a)$ is a factor of $f(x)$ iff a is a root of $f(x)$.
 - Know what the ideals are in \mathbb{Z} , \mathbb{Z}_n , $F[x]$ and $F[x]/a(x)$.

Groups

- Definitions

- group, subgroup, cyclic subgroup, abelian group;
- order of a group, order of an element;
- homomorphism, isomorphism.

- Standard examples

- The additive group of a ring.
- The group of units in a ring.
- U_n the group of units in \mathbb{Z}_n .
- $Gl(2, F)$, the group of invertible 2×2 matrices over a field F .
- $Sl(2, F)$, the group 2×2 matrices over a field F that have determinant 1.
- D_n , the group of symmetries of a regular polygon
- S_n the group of permutations of n objects.

Have a look at the last two exams and the last couple of problem sets. Here are a few extra problems:

1. Let I and J be ideals in a ring R (commutative with identity).

(a) What is $I + J$?

(b) Show that $I + J$ is an ideal.

2. Express in the simplest form.

$\langle x^2 - 1 \rangle + \langle x^2 + 2x + 1 \rangle$ in $\mathbb{Q}[x]$.

$\langle x^2 - 1 \rangle \cap \langle x^2 + 2x + 1 \rangle$ in $\mathbb{Q}[x]$.

$\langle x^2 - 1 \rangle + \langle x^2 + 2x + 1 \rangle$ in $\mathbb{Q}[x]/(x^3 - 1)$.

$\langle x^2 - 1 \rangle \cap \langle x^2 + 2x + 1 \rangle$ in $\mathbb{Q}[x]/(x^3 - 1)$.

4. Show that the function $\phi : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ defined by $\phi(f(x)) = f(2)$ is a homomorphism. (If $f(x) = \sum_{i=1}^n f_i x^i$ then $f(2) = \sum_{i=1}^n f_i 2^i$.)

What is the kernel of ϕ ?

Identify an isomorphism between $Gl(2, \mathbb{Z}_2)$ and S_3 . How many isomorphisms are there? List all the subgroups of S_3

5. Show that the set of 2×2 matrices with determinant ± 1 is a subgroup of $Gl(2, F)$.

6. Show that U_{11} is a cyclic group, generated by 2.