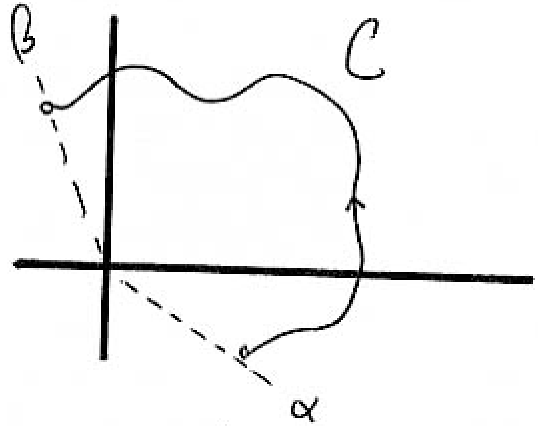
Arc length of a curve given in polar coordinates

Let $r = f(\theta)$ for $\alpha \le \theta \le \beta$.



To find the arc length of the segment, use

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$ as in Cartesian coordinates, but

write it as

$$\Delta s = \sqrt{\left(\frac{\Delta x}{\Delta\theta}\right)^2 + \left(\frac{\Delta y}{\Delta\theta}\right)^2}\ \Delta\theta \longrightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$$

Using $x = r\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta$

$y = r\sin\theta \Rightarrow \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta$

$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

This gives $\boxed{s = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta}$