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Periodic Solutions for Certain Protein Synthesis Models*

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In this paper, we investigate the qualitative behavior of a class of deterministic models with delays that arise in the study of protein synthesis. We demonstrate the existence of nonconstant periodic solutions to our system of differential-delay equations under certain conditions on the parameters. We conclude with a brief discussion of the biological significance of our results.

1. INTRODUCTION

In this paper we shall analyze a class of mathematical models for the control of protein synthesis in biosynthetic pathways. The models are based on the Jacob–Monod hypothesis [12, 16] for gene regulation in prokaryotic cells. Mathematical models of this type were first proposed by Goodwin [7, 8] and have subsequently led to numerous investigations of both experimental and theoretical nature. For a review of the biochemical model and a list of related references the reader is referred to Banks and Mahaffy [3].

Briefly, we are concerned with negative feedback or *repressible* systems such as those frequently found in biosynthetic pathways of prokaryotic organisms. In the mathematical models, one assumes that the endproduct, x_n, is fed back negatively, affecting the production of the first substance, x_1, in a nonlinear manner. The other substances, x_i, are produced linearly at a rate proportional to x_{i-1}. All substances are assumed to be degraded at a rate proportional to their own concentration. The n-dimensional generalization of Goodwin's model [7], is given by the following system of ordinary differential equations:

$$\begin{aligned}\dot{x}_1(t) &= \frac{a}{1 + kx_n(t)} - bx_1(t)\\ \dot{x}_i(t) &= \alpha_i x_{i-1}(t) - \beta_i x_i(t), \qquad i = 2,\dots, n.\end{aligned} \tag{1.1}$$

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Goodwin suggested that by introducing delays into the system to account for particular biological processes that the resulting system might exhibit sustained oscillations and be used to support the hypothesis that protein synthesis is involved in epigenetic oscillations. Banks and Mahaffy [2] showed that even with delays, the equilibrium for (1.1) is globally asymptotically stable with respect to positive initial data. In this paper, we are interested in the qualitative features of models with a higher order feedback of the form $[a/(1 + kx_n^\rho)]$. We shall give rigorous arguments to show the existence of oscillations and periodic solutions (for certain parameter values) of delay differential equations including a nonlinearity of this type.

We analyze our system of delay differential equations by using the solution operator to establish a completely continuous map of a cone into itself. By showing that the equilibrium point of our system of differential equations is ejective we are then able to use a fixed point theorem of Nussbaum [18] to prove the existence of periodic solutions. We conclude with a brief discussion of the biological implications of our results.

2. Mathematical Background

In this section we want to consider the n-dimensional Goodwin-type model with ρ repressor subunits and delays. This model can be expressed by the following system of equations:

$$\begin{aligned} \dot{x}_1(t) &= \frac{a}{1 + k[x_n(t-\tau_1) + \bar{x}_n]^\rho} - b[x_1(t) + \bar{x}_1] \\ \dot{x}_i(t) &= \alpha_i x_{i-1}(t - \tau_i) - \beta_i x_i(t), \qquad i = 2, \ldots, n, \end{aligned} \tag{2.1}$$

with the positive constants a, b, k, α_i, β_i, τ_i where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$ is the unique equilibrium solution of

$$\begin{aligned} \dot{z}_1(t) &= \frac{a}{1 + k[z_n(t-\tau_1)]^\rho} - bz_1(t) \\ \dot{z}_i(t) &= \alpha_i z_{i-1}(t - \tau_i) - \beta_i z_i(t), \qquad i = 2, \ldots, n. \end{aligned}$$

An der Heiden [1] has shown that by using the following substitutions in (2.1),

$$\begin{aligned} \hat{x}_1(t) &= x_1(t) \\ \hat{x}_i(t) &= x_i\left(t + \sum_{j=2}^{i} \tau_j\right), \qquad i = 2, \ldots, n, \end{aligned}$$

one obtains an equivalent system with only one delay $r = \sum_{i=1}^{n} \tau_i$. For notational convenience we drop the ^ and obtain the following:

$$\begin{aligned} \dot{x}_1(t) &= \frac{a}{1 + k[x_n(t-r) + \bar{x}_n]^\rho} - b[x_1(t) + \bar{x}_1] \\ \dot{x}_i(t) &= \alpha_i x_{i-1}(t) - \beta_i x_i(t), \qquad i = 2,\ldots, n. \end{aligned} \tag{2.2}$$

Define $f(\xi) = -b\bar{x}_1 + a/(1 + k(\xi + \bar{x}_n)^\rho)$.

We shall need to consider the linearization of (2.2) about the solution $x(t) \equiv 0$. This linearization has the form

$$\dot{y}(t) = Ay(t) + By(t-r) \tag{2.3}$$

where A and B are $n \times n$ constant matrices with

$$A = \begin{pmatrix} -b & 0 & \cdots & \cdots & 0 \\ \alpha_2 & -\beta_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & -\beta_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \cdots & 0 & f'(0) \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$f'(0) = -a\rho k\bar{x}_n^{\rho-1}/[1 + k\bar{x}_n^{\rho}]^2. \tag{2.4}$$

The formal adjoint equation for (2.3) is

$$\dot{\tilde{y}}(s) = -\tilde{y}(s)A - \tilde{y}(s+r)B. \tag{2.5}$$

To obtain the characteristic equation for (2.3) we consider

$$\det[A - \lambda I + Be^{-\lambda r}] = 0.$$

This is equivalent to

$$\det \begin{pmatrix} -b-\lambda & 0 & \cdots & 0 & e^{-\lambda r} f'(0) \\ \alpha_2 & -\beta_2 - \lambda & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n & -\beta_n - \lambda \end{pmatrix} = 0.$$

After expanding by the first row and using properties of the determinant of the remaining triangular matrices we see that the above becomes

$$(-b-\lambda)\left[\prod_{j=2}^{n}(-\beta_j-\lambda)\right]+e^{-\lambda r}f'(0)(-1)^{n-1}\prod_{j=2}^{n}\alpha_j$$

$$=(-1)^n\left\{(b+\lambda)\left[\prod_{j=2}^{n}(\beta_j+\lambda)\right]-\left(\prod_{j=2}^{n}\alpha_j\right)f'(0)\,e^{-\lambda r}\right\}=0.$$

Therefore, the characteristic equation for (2.3) is

$$(b+\lambda)\left[\prod_{j=2}^{n}(\beta_j+\lambda)\right]-\left(\prod_{j=2}^{n}\alpha_j\right)f'(0)\,e^{-\lambda r}=0. \tag{2.6}$$

For $n = 1$ Hadeler and Tomiak [9] and for $n = 2$ an der Heiden [1] have established conditions for existence of periodic solutions of (2.2). We shall show that under certain conditions oscillatory phenomena and periodic solutions exist for the equations (2.2) for arbitrary finite n. The technique we shall use is similar to that used by Somolinos in [19]. First, we need to define an ejective point of a map $\mathscr{T}$.

DEFINITION 2.1. Suppose X is a Banach space, U is a subset of X, and x is a given point of U. Given a map $\mathscr{T}\colon U\backslash\{x\}\to X$, the point $x\in U$ is said to be an *ejective point of* $\mathscr{T}$ if there is an open neighborhood $G\subseteq X$ of x such that for every $y\in G\cap U$, $y\neq x$, there is an integer $m=m(y)$ such that $\mathscr{T}_y^m\notin G\cap U$.

To show existence of periodic solutions of (2.2) the main result we shall make use of is the following theorem:

THEOREM 2.1 (Nussbaum [18]). *If K is a closed, bounded, convex, ∞-dimensional set in a Banach space X, $\mathscr{T}\colon K\backslash\{x_0\}\to K$ is completely continuous, and $x_0\in K$ is an ejective point of $\mathscr{T}$, then there is a fixed point of $\mathscr{T}$ in $K\backslash\{x_0\}$.*

To use this theorem we need some way of demonstrating ejectivity. First, we must define some terminology that we shall use and then state a theorem of Chow and Hale [5].

DEFINITION 2.2. Let $\mathscr{C}=\mathscr{C}([-r,0];R^n)$ be the Banach space of continuous functions mapping $[-r,0]$ into R^n with the topology of uniform convergence. Let $y_t\in\mathscr{C}$ denote $y(t+\theta)$ for $\theta\in[-r,0]$. For any characteristic root λ of (2.3) there is a decomposition of $\mathscr{C}$ as $\mathscr{C}=P_\lambda\oplus Q_\lambda$, where P_λ and Q_λ are invariant under the solution operator $\mathscr{T}_L(t)$ of (2.3), $\mathscr{T}_L(t)\phi=y_t(\phi)$, $\phi\in\mathscr{C}$. For further details about this decomposition see Hale [10, p. 168 ff]. We define the projection operators of the above decomposition of $\mathscr{C}$ by π_λ with range equal to P_λ and $I-\pi_\lambda$ with range equal to Q_λ.

To show ejectivity we shall make use of a theorem of the following type:

THEOREM 2.2 (Chow–Hale [5]). *Suppose the following conditions are fulfilled.*

(i) *There is a characteristic root* λ *of* (2.3) *satisfying* $\operatorname{Re}\lambda > 0$.

(ii) *There is a convex set* $\bar{K} \subseteq \mathscr{C}$, $0 \in \bar{K}$, *and* $\delta > 0$, *such that*

$$v = v(\delta) \equiv \inf\{\| \pi_\lambda \psi \| : \psi \in \bar{K}, \| \psi \| = \delta\} > 0.$$

(iii) *There is a completely continuous function* τ: $\bar{K}\backslash\{0\} \to [\alpha, \infty)$, $0 \leqslant \alpha$ *such that the function defined by*

$$\mathscr{T}\psi = x_{\tau(\psi)}(\psi), \qquad \psi \in \bar{K}\{0\},$$

where x_t *is a solution of* (2.2), *maps* $\bar{K}\backslash\{0\}$ *into* $\bar{K}$ *and is completely continuous.*

Then $0 \in \mathscr{C}$ *is an ejective point of* $\mathscr{T}$.

For our protein synthesis model we shall actually need to prove and use a variation of this theorem in order to demonstrate ejectivity in a space slightly different from $\mathscr{C}$.

3. Analysis of the Characteristic Equation

At this point we shall derive conditions under which at least two roots of (2.6) lie in the right half of the complex plane. In fact, we shall also show these roots lie in a strip in the complex plane between the lines $z = \pm(\pi/r)i$. These bounds will be used later in verification of condition (ii) of our analog to Theorem 2.2. Specifically, employing the argument principle, we shall analyze the image of the closed contour Γ under the transformation $F(\lambda)$ where

$$F(\lambda) \equiv (b + \lambda)\left[\prod_{j=2}^{n} (\beta_j + \lambda)\right] - \left(\prod_{j=2}^{n} \alpha_j\right) f'(0)\, e^{-\lambda r}$$
$$= D(\lambda)\, e^{i\theta(\lambda)} + C_1 e^{-\lambda r},$$

with $C_1 = -(\prod_{j=2}^{n} \alpha_j) f'(0) > 0$, a constant, and where Γ (except in special cases) is as diagrammed in Fig. 3.1. Let $\lambda = \mu + i\nu$, then by setting $(b + \lambda) = [\nu^2 + (b + \mu)^2]^{1/2} e^{i\theta_1(\lambda)}$, where $\theta_1(\lambda) = \arctan(\nu/(b + \mu))$, and doing the same with the $\beta_j + \lambda$, we see that

$$D(\lambda) = [\nu^2 + (b + \mu)^2]^{1/2} \prod_{j=2}^{n} [\nu^2 + (\beta_j + \mu)^2]^{1/2}$$

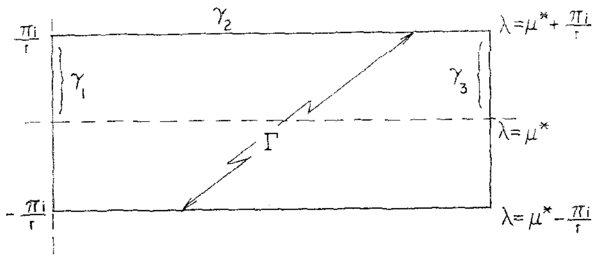

FIG. 3.1. The contour Γ.

and

$$\theta(\lambda) = \arctan\left(\frac{\nu}{b+\mu}\right) + \sum_{j=2}^{n} \arctan\left(\frac{\nu}{\beta_j+\mu}\right).$$

For convenience let us further define

$$D_1(\lambda) \equiv D(\lambda)e^{i\theta(\lambda)}.$$

In our analysis we shall compare the orientation of F and D_1 with respect to the origin and determine the number of clockwise encirclements of the origin by F as λ moves along Γ. Observe that we have alignment of F, D_1, and the origin whenever (see Fig. 3.2)

$$\arg[F(\lambda) - D_1(\lambda)] = -k\pi + \arg D_1(\lambda), \text{ for some integer } k,$$

or

$$\theta(\lambda) - \arg[F(\lambda) - D_1(\lambda)] = \theta(\lambda) + r\nu = k\pi. \tag{3.1}$$

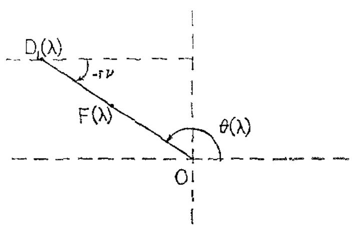

FIG. 3.2. Alignment condition.

Along γ_1, $-\pi \leqslant \arg(F - D_1) \leqslant 0$ and $\theta(\lambda) \geqslant 0$, therefore k is a non-negative integer. Clearly at $\lambda = 0$ we have alignment as both $D_1(0)$ and $F(0)$ lie on the positive real axis. This corresponds to $k = 0$. As ν increases along γ_1, $\theta(\lambda)$ increases and $\arg(F - D_1)$ decreases. Since at $\nu = \pi/r$, $\arg[F(i\pi/r) -$

$D_1(i\pi/r)] = -\pi$, this implies $\theta(i\pi/r) - \arg[F(i\pi/r) - D_1(i\pi/r)] > \pi$; hence there exists a ν_0, $0 < \nu_0 < \pi/r$, such that

$$\theta(i\nu_0) - \arg[F(i\nu_0) - D_1(i\nu_0)] = \theta(i\nu_0) + r\nu_0 = \pi.$$

This will be the first alignment after $\lambda = 0$. Now if they exist define ν_k as the successive alignments of D_1, F and the origin as λ moves along γ_1 from 0 to $i\pi/r$ and such that

$$\theta(i\nu_k) - \arg[F(i\nu_k) - D_1(i\nu_k)] = \theta(i\nu_k) + r\nu_k = (k+1)\pi, \quad 1 \leqslant k \leqslant k_0, \tag{3.2}$$

where ν_{k_0} denotes the last alignment along γ_1. Note that from the formula for $\theta(\lambda)$ we see that $(k+1)\pi < n\pi/2$.

Let μ^* be chosen arbitrarily large and let λ traverse γ_2 where along γ_2, $\theta(\lambda)$ decreases monotonically toward zero. Using (3.1) and matching the indices with (3.2) we can ennumerate the alignments along γ_2 (if they exist) by the formula

$$\theta(\lambda_k) - \arg[F(\lambda_k) - D_1(\lambda_k)] = \theta(\lambda_k) + \pi = (k+1)\pi$$

or

$$\theta(\lambda_k) = k\pi \qquad \text{for} \quad 1 \leqslant k \leqslant k_0. \tag{3.3}$$

Notice that the first λ_k is λ_{k_0} with the last being λ_1, that is $\operatorname{Re}\lambda_k < \operatorname{Re}\lambda_{k-1}$. There is the possibility that alignment occurs at $\lambda = i\pi/r$ as a special case. In this special case we define $i\nu_{k_0} = \lambda_{k_0}$. This special case can be handled the same as the other cases in our arguments below.

Along γ_3, $\theta(\lambda)$ decreases to zero with $\theta(\mu^*) = 0$. D_1, F and the origin align at $\lambda = \mu^*$ since $D_1(\mu^*)$ and $F(\mu^*)$ lie on the real axis. Now we shall use the above information to prove the following:

PROPOSITION 3.1. (i) *Suppose* $D(i\nu_0) > C_1$, *then* $F(\lambda)$ *does not encircle the origin as* λ *goes around* Γ, *hence by the argument principle no roots of* $F(\lambda) = 0$ lie inside Γ.

(ii) *Suppose* $D(i\nu_0) < C_1$, *then* $F(\lambda)$ *encircles the origin as* λ *goes around* Γ. *The argument principle can be used to demonstrate that at least two roots of* $F(\lambda) = 0$ *lie inside* Γ.

Remark. If $D(i\nu_0) = C_1$, then $F(i\nu_0) = 0$ and so the image of Γ under the transformation F passes through the origin. This means that the zeros of F lie on the imaginary axis at $\pm i\nu_0$.

(This will be the case when a Hopf bifurcation for one of the parameters such as r occurs.)

Proof. Our argument will compare $F(\lambda)$ to $D_1(\lambda)$ to determine the orientation of F with respect to the origin. Geometrically, it is easily seen that $D_1(\lambda)$ winds counterclockwise as λ goes along γ_1 then unwinds as λ goes along γ_2 and γ_3, giving a net of no encirclements which was to be expected since all zeros of D_1 lie in the left half plane.

Case (i). (see Fig. 3.3) Suppose $D(i\nu_0) > C_1$. In this case at ν_0, $F(i\nu_0)$ lies between $D(i\nu_0)$ and the origin. Now note that along γ_1 and γ_2 clockwise, $D(\lambda)$ is strictly increasing as is easily seen from the formula for $D(\lambda)$. For each ν_k, k odd we have $D_1(i\nu_k)$ lying between the origin and $F(i\nu_k)$. Since $D(\lambda)$ is increasing, for ν_k, k even $F(i\nu_k)$ is lying between $D_1(i\nu_k)$ and the origin. Similarly along γ_2 when k is odd $D_1(\lambda_k)$ lies between the origin and $F(\lambda_k)$, and for k even $F(\lambda_k)$ lies between the origin and $D_1(\lambda_k)$. Thus $F(\lambda)$ has the same orientation relative to the origin as $D_1(\lambda)$, hence the image of Γ under the transformation F does not encircle the origin. (To encircle or have a different orientation we must have the origin between $F(\lambda)$ and $D_1(\lambda)$ at some λ.)

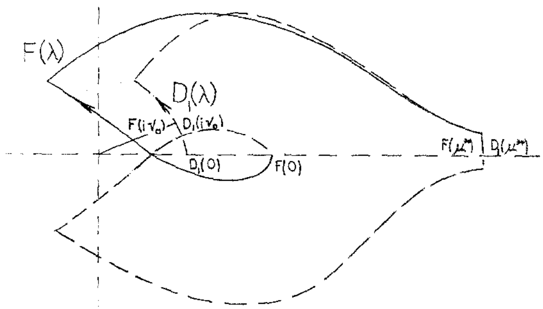

FIG. 3.3. No encirclement.

Case (ii). (see Fig. 3.4) Suppose $D(i\nu_0) < C_1$. Since $D(\lambda)$ is increasing along γ_1 and γ_2 clockwise with $D(\lambda) \to +\infty$ as $\mu^* \to +\infty$, for μ^* sufficiently large, there exists λ^* such that $D(\lambda^*) = C_1$. If $\lambda^* = i\nu_{2k}$ or λ_{2k}, $k = 1, 2,\ldots$, then we must modify Γ. In these situations we see that a zero of F lies on Γ (another bifurcation is occurring if $\lambda^* = i\nu_{2k}$), hence more roots of $F(\lambda) = 0$ are passing into the region enclosed by Γ. Since we only need to find two zeros of F inside Γ we are not particularly interested in these roots and so can modify Γ by making an ϵ-radius semicircle on Γ about λ^* to exclude this point and thus not obtain encirclement due to these zeros of F.

Let $D(\lambda^*) = C_1$ (or in the above special cases where we modify Γ as stated above and let $D(\lambda^*) = C_1$ where now λ^* will be on that ϵ-radius semicircle). Going clockwise on Γ from 0 to λ^*, $D(\lambda) < C_1$; and from λ^* to μ^*, $D(\lambda) > C_1$, which will be just as in Case (i) and so F will have the same orientation relative to the origin as D_1 on this part of Γ.

First we handle the case when only ν_0 exists with $D(i\nu_0) < C_1$, so that alignment occurs only at $\lambda = 0$, $\pm i\nu_0$, and μ^*. As λ goes from 0 to $i\nu_0$, arg $D_1(\lambda)$

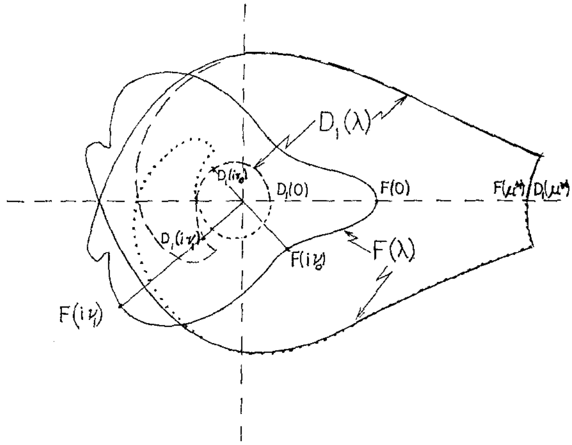

FIG. 3.4. Two encirclements.

increases monotonically from 0 to $\theta(i\nu_0)$. At $i\nu_0$ the origin lies between $D_1(i\nu_0)$ and $F(i\nu_0)$, i.e., $\arg F(i\nu_0) = \theta(i\nu_0) \pm \pi$. Here $\arg F(i\nu_0) = \theta(i\nu_0) - \pi$ since $\arg F(i\nu_0) = \theta(i\nu_0) + \pi$ implies there exists a $\tilde{\nu}$, $0 < \tilde{\nu} < \nu_0$, such that $\arg F(i\tilde{\nu}) = \theta(i\tilde{\nu})$ which contradicts the definition of ν_0. As λ proceeds from $i\nu_0$ to $i\pi/r$ along Γ we must show that $F(i\nu)$ does not have the same orientation as D_1. If F were to have the same orientation as D_1 then F must go counterclockwise about the origin which means there exists a $\hat{\nu}$ with $\nu_0 < \hat{\nu} < \pi/r$ such that $\operatorname{Im} F(i\hat{\nu}) = 0$ and $\operatorname{Re} F(i\hat{\nu}) > 0$. At $i\hat{\nu}$, $\theta(i\hat{\nu}) > \theta(i\nu_0)$ by monotonicity, while $\arg[F(i\hat{\nu}) - D_1(i\hat{\nu})] = -\hat{\nu}r$. Since $-\pi \leqslant \arg[F - D_1] \leqslant 0$, we have $\operatorname{Im} D_1(i\hat{\nu}) \geqslant \operatorname{Im} F(i\hat{\nu}) = 0$ so that $\theta(i\hat{\nu}) < \pi$. We observe that since $F(i\hat{\nu})$ lies on the positive real axis and $\theta(i\hat{\nu}) < \pi$ we have $\arg[F(i\hat{\nu}) - D_1(i\hat{\nu})] > \theta(i\hat{\nu}) - \pi$. Therefore, we see that $-\hat{\nu}r > \theta(i\hat{\nu}) - \pi > \theta(i\nu_0) - \pi = -v_0 t$ which is a contradiction of $\hat{\nu} > \nu_0$. Thus, we see that for $\nu_0 < \nu < \pi/r$ along γ_1, F has an orientation relative to the origin which is opposite to D_1. Since we are assuming only ν_0 exists there are no λ_k's so from (3.3) we see that $\theta(\lambda) < \pi$ or $\operatorname{Im} D_1(\lambda) > 0$ along γ_2. At $\lambda = i\pi/r$, $\arg F(i\pi/r) = \theta' - 2\pi$, while $\theta(i\pi/r) = \bar{\theta}$ where $\bar{\theta}, \theta' < \pi$. Along γ_2, $\operatorname{Im} F(\lambda) = \operatorname{Im} D_1(\lambda)$ and $F \to D_1$ exponentially since $F(\lambda) = D_1(\lambda) - C_1 e^{-\mu r}$. Let $\mu^* \to +\infty$, then $\arg F \to 0$ and $|F| \to +\infty$ which implies that relative to the origin $\arg F(\lambda)$ has gone -2π radians as λ has gone from 0 to μ^* along γ_1, γ_2, and γ_3. By using a symmetry argument we see that $F(\lambda)$ encircles the origin twice as λ goes around Γ.

Now suppose ν_1 exists. Again since $D(i\nu_0) < C_1$ at $i\nu_0$, D_1 has traversed $\theta(i\nu_0)$ radians while F has gone $\theta(i\nu_0) - \pi$ radians. At $\lambda = i\nu_1$ we have $D_1(i\nu_1)$ between the origin and $F(i\nu_1)$. Following a similar argument to the one when only ν_0 existed we find that $D_1(i\nu_1)$ will have traversed $\theta(i\nu_1)$ radians while $\arg F(i\nu_1) = \theta(i\nu_1) - 2\pi$ which implies $F(i\nu_1)$ has gone clockwise one revolution relative to the origin compared to $D_1(i\nu_1)$.

We continue in the same manner. Suppose ν_2 exists and $\operatorname{Im} \lambda^* > \nu_2$, then at $i\nu_2$ we have the origin between $F(i\nu_2)$ and $D_1(i\nu_2)$ (since $D(i\nu_2) < C_1$). If ν_3 exists, then $D_1(i\nu_3)$ lies between the origin and $F(i\nu_3)$. $\operatorname{Arg} F(i\nu_3) = \theta(i\nu_3) - 4\pi$ and $\arg D_1(i\nu_3) = \theta(i\nu_3)$ which make a difference of two clockwise encirclements relative to the origin using similar arguments to those above with ν_0 and ν_1.

Now continue in the same spirit until either (a) $\operatorname{Im} \lambda^* < \nu_{2j}$ for some $j = 1, 2, \ldots$ or (b) the ν_k's are exhausted. If, in case (a) $\operatorname{Im} \lambda^* < \nu_{2j_0}$ for some j_0 ($\operatorname{Im} \lambda^* > \nu_{2j_0-2}$), then along Γ from $i\nu_{2j_0}$ to μ^* the orientation of F is the same as D_1 as in Case (i). Arguing as above, along γ_1 from 0 to $i\nu_{2j_0}$ we find that $\arg D_1(i\nu_{2j_0-1}) = \theta(i\nu_{2j_0-1})$ and $\arg F(i\nu_{2j_0-1}) = \theta(i\nu_{2j_0-1}) - 2j_0\pi$, which gives a difference of j_0 clockwise rotations relative to the origin. $D_1(\lambda)$ does not encircle the origin which implies that relative to the origin along $\gamma_1, \gamma_2, \gamma_3$, $F(\lambda)$ has j_0 clockwise rotations about the origin, so again from symmetry we see that $F(\lambda)$ has $2j_0$ clockwise encirclements of the origin as λ traverses Γ.

If on the other hand, in case (b) the ν_k's are exhausted (after $k \geqslant 1$) then there exists λ_{k_0} such that $\theta(\lambda_{k_0}) = k_0\pi$ since $D_1(\lambda)$ does not encircle the origin as λ traverses Γ along $\gamma_1, \gamma_2, \gamma_3$. Let j_0 be even with λ_{j_0} being the first λ_j, j even, along γ_2 such that $D(\lambda_{j_0}) > C_1 e^{-\mu_{j_0} r}$ (take $j_0 \equiv 0$ if $D(\lambda_j) < C_1 e^{-\mu_j r}$ for all λ_j on γ_2). From λ_{j_0} to μ^* again we see that relative to D_1, F behaves as in Case (i) and so $F(\lambda)$ has the same orientation as $D_1(\lambda)$ relative to the origin. In the special case $j_0 = 0$ take λ_0 such that $\operatorname{Re} \lambda_0 > \operatorname{Re} \lambda_1$ on γ_2 and the above holds.

Recall $\arg[F - D_1] = -\pi$ on γ_2 so whenever j is odd $D_1(\lambda_j)$ lies between $F(\lambda_j)$ and the origin, $\theta(\lambda_j) = j\pi$, i.e., F and D_1 are on the negative real axis. Whenever j is even $j > j_0$, the origin lies between $F(\lambda_j)$ and $D_1(\lambda_j)$, i.e., $F(\lambda_j) < 0$ and $D_1(\lambda_j) > 0$ both real, while for j even and $j \leqslant j_0$, $F(\lambda_j)$ and $D_1(\lambda_j)$ lie on the positive real axis with $F(\lambda_j) < D_1(\lambda_j)$. Along γ_2 until $\lambda = \lambda_{j_0}$, $F(\lambda)$ never crosses $\arg F(\lambda) = 2m\pi$ for any $m \in \mathbf{Z}$, thus F has no encirclements of the origin while $D_1(\lambda)$ encircles the origin clockwise each time $D_1(\lambda_{2j})$ goes to $D_1(\lambda_{2j-2})$ as λ goes from λ_{2j} to λ_{2j-2} along along Γ (provided these do exist). From this we can now evaluate the number of encirclements of the origin by F in case (b).

First suppose k_0 is even; then along γ_1, $D_1(i\nu)$ traverses $(k_0/2 + \tilde{\theta}/(2\pi)) \cdot 2\pi$ radians while $F(i\nu)$ goes $(\hat{\theta}/(2\pi) - 1) \cdot 2\pi$ radians where $\tilde{\theta} \equiv \theta(i\pi/r) \bmod 2\pi$ and $\hat{\theta} \equiv \arg F(i\pi/r) \bmod 2\pi$. Along γ_2 and γ_3, $D_1(\lambda)$ goes $-(k_0/2 + \tilde{\theta}/(2\pi)) \cdot 2\pi$ radians since D_1 does not encircle the origin, while F traverses $-(\hat{\theta}/(2\pi) + j_0/2) \cdot 2\pi$ radians with $j_0/2$ being the number of times that $F(\lambda)$ follows $D_1(\lambda)$ around the origin as in Case (i). Therefore, the net encirclement around the origin for F is $-(\hat{\theta}/(2\pi) + j_0/2) \cdot 2\pi + (\hat{\theta}/(2\pi) - 1) \cdot 2\pi = -(1 + j_0/2) \cdot 2\pi$ radians along $\gamma_1, \gamma_2, \gamma_3$ or $1 + j_0/2$ clockwise encirclements. Again, by using the symmetry we obtain $2 + j_0$ clockwise encirclements of the origin by $F(\lambda)$ as λ proceeds clockwise around Γ.

Now suppose k_0 is odd; then along γ_1, $D_1(i\nu)$ traverses $((k_0 - 1)/2 + \tilde{\theta}/(2\pi)) \cdot 2\pi$ radians while $F(i\nu)$ goes $(\hat{\theta}/(2\pi) - 1) \cdot 2\pi$ radians where $\tilde{\theta}$ and $\hat{\theta}$ are as before. Along γ_2 and γ_3, $D_1(\lambda)$ must return $-((k_0 - 1)/2 + \tilde{\theta}/(2\pi)) \cdot 2\pi$ radians since it does not encircle the origin, while F along γ_2 and γ_3 does not cross $\arg F = 2m\pi$, m an integer, until λ_{j_0}, then $F(\lambda)$ follows $D_1(\lambda)$, so F traverses $-(\hat{\theta}/(2\pi)+j_0/2) \cdot 2\pi$ radians. Combining our information along $\gamma_1, \gamma_2, \gamma_3$ we see that relative to $D_1(\lambda)$, $F(\lambda)$ makes an angle of $-(\hat{\theta}/(2\pi) + j_0/2) \cdot 2\pi + (\hat{\theta}/(2\pi) - 1) \cdot 2\pi = -(1 + j_0/2) \cdot 2\pi$ radians or F encircles the origin clockwise $1 + j_0/2$ times along $\gamma_1, \gamma_2, \gamma_3$. By symmetry again we see that $F(\lambda)$ encircles the origin clockwise $j_0 + 2$ times as λ winds clockwise around Γ.

Using the above information with the argument principle we have the proposition. Q.E.D.

The proof of the above proposition, in fact, shows us exactly how many roots of $F(\lambda) = 0$ lie inside Γ (if there are any). At this point we shall verify that there are cases when Proposition 3.1(ii) can be used on our Goodwin-type repression model. Consider the example when $a = 2$, $\rho = 4$, $b = k = \alpha_j = \beta_j = 1$. From (2.2) and (2.4) it is easily seen that $\bar{x} = (1,..., 1)^T$ and $f'(0) = -2$. Hence, $C_1 = 2\prod_{j=2}^{n} \alpha_j = 2$ and $D(i\nu) = (b^2 + \nu^2)^{1/2} \prod_{j=2}^{n} (\beta_j^2 + \nu^2)^{1/2} = [(1 + \nu^2)^{1/2}]^n$. So the condition $[(1 + \nu_0^2)^{1/2}]^n < 2$, which is $D(i\nu_0) < C_1$, is equivalent to $\nu_0^2 < 4^{1/n} - 1$. But $\nu_0 < \pi/r$, so we find roots of $F(\lambda) = 0$ inside Γ if $r > \pi/(4^{1/n} - 1)^{1/2}$. When $n = 2$, then $r > \pi$ will guarantee that the condition and conclusion of Proposition 3.1(ii) hold. For arbitrary n, if $b\prod_{j=2}^{n} \beta_j < -f'(0)\prod_{j=2}^{n} \alpha_j$, then since

$$(b^2 + \nu_0^2)^{1/2} \prod_{j=2}^{n} (\beta_j^2 + \nu_0^2)^{1/2} < [b^2 + (\pi/r)^2]^{1/2} \prod_{j=2}^{n} [\beta_j^2 + (\pi/r)^2]^{1/2}$$

and since $\lim_{r\to\infty} [b^2 + (\pi/r)^2]^{1/2} \prod_{j=2}^{n} [\beta_j^2 + (\pi/r)^2]^{1/2} = b\prod_{j=2}^{n} \beta_j$, we can easily see by similar calculations to the ones above that for r sufficiently large the condition and conclusion of Proposition 3.1(ii) will hold.

From the above information we have found that there exist conditions under which (2.2) is locally unstable. This is the first step in showing that periodic solutions to (2.2) may exist. We shall now demonstrate that under appropriate conditions solutions of (2.2) oscillate. From this, we shall obtain a completely continuous map needed to make use of Theorem 2.1 and an analog to Theorem 2.2. Under additional hypotheses we shall then be able to use Proposition 3.1 to establish ejectivity and hence obtain the existence of periodic solutions to (2.2).

4. Oscillations of the Goodwin-Type Model

At this point, we must define the cone with which we shall be working to utilize Theorem 2.1 and our modification of Theorem 2.2. We start by defining the cone K similar to that of an der Heiden [1].

DEFINITION 4.1. Let $\mathscr{C}_0 = R^{n-1} \times \mathscr{C}([-r, 0]; R)$ be the Banach space formed by the product of Euclidean $n - 1$ space and the space of continuous functions mapping $[-r, 0]$ into R with the usual product topology; that is, if $\psi = (\mathbf{x}_0, \phi_n) \in \mathscr{C}_0$ with $\mathbf{x}_0 = (x_1, ..., x_{n-1})^T \in R^{n-1}$ and $\phi_n \in \mathscr{C}([-r, 0]; R)$, then $\| \psi \|_{\mathscr{C}_0} = (\sum_{i=1}^{n-1} | x_i |^2 + \sup_{\theta \in [-r,0]} | \phi_n(\theta)|^2)^{1/2}$. Let $K \subseteq \mathscr{C}_0$ with $K \equiv \{\psi = (\mathbf{x}_0, \phi_n)$: $\mathbf{x}_0 \in R^{n-1}$ with $x_i \geqslant 0$, $i = 1,..., n - 1$, $\phi_n(-r) = 0$, and $e^{\beta_n t}\phi_n(t)$ nondecreasing on $[-r, 0]\}$. Now for any $\psi \in \mathscr{C}_0$ with $\psi = (\mathbf{x}_0, \phi_n)$ with $\mathbf{x}_0$ satisfying $x_i \geqslant -\bar{x}_i$ for $i < n$ and $\phi_n(\theta) \geqslant -\bar{x}_n$ for $\theta \in [-r, 0]$, one can show that (2.2) with initial data ψ has a unique solution defined on $[0, \infty)$ (Mahaffy [17]). Let $z_t \in \mathscr{C}_0$ designate a solution of (2.2) with $z_t = (x(t), x_n^t)$ where $x(t) = (x_1(t),..., x_{n-1}(t))^T$ and $x_n^t \in \mathscr{C}([-r, 0); R)$ is the function with values $x_n(t + \theta)$ for $\theta \in [-r, 0]$.

Next, we must make use of a result of Mahaffy [17]. In establishing our results for global asymptotic stability of (2.2) for certain parameter values we also established bounds for the asymptotic limit of the solution in the general case. We derive the following $\rho^2 + 1$ degree polynomial in X where X can be either an upper or lower bound on $x_1(t) + \bar{x}_1$,

$$b(b + C_2 X^\rho)^\rho X + C_2 a^\rho X - a(b + C_2 X^\rho)^\rho = 0 \tag{4.1}$$

with $C_2 \equiv kb[\prod_{j=2}^{n} (\alpha_j/\beta_j)]^\rho$. If there is a unique positive solution to (4.1) then we have global asymptotic stability; however, if there exist multiple roots then the asymptotic limit, $\lim_{t\to\infty}(x_1(t) + \bar{x}_1)$ will be bounded below by the smallest root of (4.1) greater than zero and bounded above by the largest root of (4.1) less than a/b. Similarly one obtains asymptotic bounds on $x_2(t),..., x_n(t)$. Let $U_1,..., U_n$ be the lower bounds and $V_1,..., V_n$ be the upper bounds. Since the lower bounds are greater than zero, $U_1 > 0$, we can choose an $\epsilon > 0$ such that $0 < U_1 - \epsilon \equiv U^1$(clearly $U^1 < U_1$). Let $U^i = \prod_{j=2}^{i} (\alpha_j/\beta_j)U^1$, $i = 2,..., n$. We define $V^1 = a/b[1 + k(U^n)^\rho]$ and let $V^i = (\prod_{j=2}^{i} (\alpha_j/\beta_j)]V^1$. From results of Mahaffy [17] we can see that $0 < U^i < U_i$ and $V^i > V_i$, $i = 1,..., n$; hence, if we start with initial conditions in K then after some finite time t^* the solution of (2.2) has the following bounds

$$U^i - \bar{x}_i \leqslant x_i(t) \leqslant V^i - \bar{x}_i, \qquad i = 1,..., n.$$

Define the above region in R^n described by these inequalities as $\mathscr{R}$. By the definition of the region $\mathscr{R}$ it is easily seen from (2.2) that if the initial data is such that $z(t) \in \mathscr{R}$ for $t \in [-r, 0]$, then $z(t) \in \mathscr{R}$ for all $t \geqslant 0$.

We are now ready to define the cone K_0 that we shall use in proving the existence of oscillations and periodic solutions of (2.2). Let

$$K_0 = \{\psi = (\mathbf{x}_0, \phi_n): \psi \in K, (\mathbf{x}_0, \phi_n(0)) \in \mathscr{R} \text{ and for all } t \in [-r, 0], 0 \leqslant \phi_n(t) \leqslant V^n - \bar{x}_n\}.$$

To identify the position of the solution $z(t) \in R^n$ at time t, we shall adopt a notation similar to the one that Hastings, *et al.* [11] use in their paper which deals with the case when $r = 0$, i.e., a system of ordinary differential equations. In identifying the position of $z(t) = (x_1(t),..., x_n(t))^T$ we shall divide R^n into its orthants using the notation $S(b_1,..., b_n)$ where the b_i's equal either 0 or 1. $b_i = 0$ will denote that the ith coordinate is negative while $b_i = 1$ will represent nonnegativity of the ith coordinate. For example, by starting with initial conditions in K we see that $x(0) \in S(1,..., 1)$; that is $x_i(0)$ is nonnegative for each $i = 1,..., n$. We shall show that the solution of (2.2) follows the same direction as in the ordinary differential equation case of Hastings, Tyson, and Webster. Starting with $x(0) \in S(1,..., 1)$ as t increases $z(t)$ proceeds to $S(0, 1,..., 1)$, then into $S(0, 0, 1,..., 1)$, etc. until $z(t) \in S(0,..., 0)$. $z(t)$ then passes into $S(1, 0,..., 0)$, then into $S(1, 1, 0,..., 0)$, etc., returning to $S(1,..., 1)$ and completing a cycle.

We shall now follow the trajectory through one cycle and then by showing that the solution is back where we started we can demonstrate oscillations of the system (2.2). To analyze this cycle we shall use the following formulae derived from (2.2) using the variation of constants formula:

$$\begin{aligned} x_1(t) &= \int_{\bar{t}}^{t} \left[-b\bar{x}_1 + \frac{a}{1 + k[x_n(s-r) + \bar{x}_n]^\rho}\right] e^{-b(t-s)}\, ds + x_1(\bar{t})\, e^{-b(t-\bar{t})} \\ &= \int_{\bar{t}}^{t} f(x_n(s-r))\, e^{-b(t-s)}\, ds + x_1(\bar{t})\, e^{-b(t-\bar{t})}, \end{aligned} \tag{4.2a}$$

$$x_i(t) = \int_{\bar{t}}^{t} \alpha_i x_{i-1}(s)\, e^{-\beta_i(t-s)}\, ds + x_i(\bar{t})\, e^{-\beta_i(t-\bar{t})}. \tag{4.2b}$$

We shall also need a hypothesis on the nonlinear term f.

(H1) Assume that the nonlinear term $f(\xi)$ satisfies the following: There exist a $\delta > 0$ and $c > (b/\boldsymbol{\alpha})(e^{\boldsymbol{\beta} r} - 1)^{-1}$ with $\boldsymbol{\alpha} \equiv \prod_{j=2}^{n} (\alpha_j/\beta_j)$ and $\boldsymbol{\beta} \equiv \min\{b, \beta_2,..., \beta_n\}$ such that $|f(\xi)| \geqslant c\,|\xi|$ for $|\xi| \leqslant \delta$.

We must show that our f satisfies (H1). Choose c positive with $c < a\rho k\bar{x}_n^{\rho-1}/[1 + k\bar{x}_n{}^\rho]^2$. Now we consider the Maclaurin series expansion of f,

$$f(\xi) = f(0) + f'(0)\xi + f''(0)\xi^2 + \cdots = f'(0)\xi + f''(0)\xi^2 + \cdots, \tag{4.3}$$

where the coefficient of the linear term $f'(0)$ is given by (2.4). From this we see $c < f'(0)$. Now take $0 < r < +\infty$, large enough so that $(b/\boldsymbol{\alpha})(e^{\boldsymbol{\beta} r} - 1)^{-1} < c$. For ξ sufficiently near the origin the higher order terms are negligible and so the linear term of f dominates. This implies there exists $\delta = \delta(c) > 0$ such that $|f(\xi)| \geqslant c\,|\xi|$ for $|\xi| \leqslant \delta$. So we observe that for r sufficiently large, where r depends on our choice of c, f satisfies (H1).

THEOREM 4.1. *Assume* (H1). *Starting with $\psi \in K_0\backslash\{0\}$ as initial conditions for* (2.2), *then for z_t the corresponding solution of* (2.2),

(i) *there exists a continuous function* $\tau_1(\psi)$: $K_0\backslash\{0\} \to (r, \infty)$ *such that* $z_{\tau_1(\psi)}(\psi) \in K_1\backslash\{0\}$ *where* $K_1 \equiv \{\psi = (\mathbf{x}_0, \phi_n): -\psi \in K, (\mathbf{x}_0, \phi_n(0)) \in \mathscr{R}$ *and for all* $t \in [-r, 0]$, $U^n - \bar{x}_n \leqslant \phi_n(t) \leqslant 0\}$,

(ii) *there exists a continuous function* $\tau_2(\psi)$: $K_1\backslash\{0\} \to (r, \infty)$ *such that* $z_{\tau_2(\psi)}(\psi) \in K_0\backslash\{0\}$,

(iii) $z(t)$ *is oscillatory, i.e., there exists a sequence of times* $\{s_k\}_{k=1}^{\infty}$ *with* $|s_k - s_{k-1}| > r$ *such that* $x_1(s_k) = 0$ *and* $x_i(s_k) \geqslant 0$ *(not all* $= 0$), $i = 2,..., n$, *for* $k = 1 \bmod 2$ *and* $x_i(s_k) \leqslant 0$, $i = 2,..., n$, *for* $k = 0 \bmod 2$.

Proof. To prove this we shall prove (i). A very similar proof will establish (ii). Using the proofs of (i) and (ii), we can argue (iii) quite easily. To show (i) we need a series of lemmas.

Using the above notation we start with initial data $\psi_1 \in K_0\backslash\{0\}$ and so $z(0) \in S(1,..., 1)$. Suppose at some t_0, $x_i(t_0) = 0$ for some $i \geqslant 2$. Now since $x_{i-1}(t_0) \geqslant 0$ this implies that $\dot{x}_i(t_0) \geqslant 0$ which implies that the trajectory $z(t)$ either remains in $S(1,..., 1)$ or it leaves $S(1,..., 1)$ through $x_1 = 0$ and passes into $S(0, 1,..., 1)$. We can assume that for $t \in [0, t_0)$ for some t_0 that the trajectory is in $S(1,..., 1) \cap \{x_1 > 0\}$. Then since $x_n(t) \geqslant 0$ for $t \geqslant -r$ it is seen that

$$\dot{x}_1(t) = \frac{a}{1 + k[x_n(t-r) + \bar{x}_n]^\rho} - b[x_1(t) + \bar{x}_1]$$
$$\leqslant \frac{a}{1 + k\bar{x}_n{}^\rho} - b\bar{x}_1 - bx_1(t) = -bx_1(t) < 0.$$

Therefore, $x_1(t)$ is strictly decreasing for $z(t) \in S(1,..., 1) \cap \{x_1 > 0\}$, $0 \leqslant t < t_0$, and in fact $x_1(t) \to 0$; say $x_1(t_0) = 0$. We shall show that t_0 is finite. First we shall investigate what happens when t_0 is infinite.

LEMMA 4.1. *Suppose* $x_1(t)$ *is strictly decreasing to zero and we remain in* $S(1,..., 1)$. *Then after some finite time* T_0, $\dot{x}_i(t) < 0$ *for all* $t \geqslant T_0$, $i = 1,..., n$, *i.e.*,

$$x_n(t) > (\alpha_n/\beta_n)\, x_{n-1}(t) > \cdots > \left(\prod_{j=2}^{n} (\alpha_j/\beta_j)\right) x_1(t) = \boldsymbol{\alpha} x_1(t)$$

for all $t \geqslant T_0$. *Furthermore,* $x_i(t) \searrow 0$ *as* $t \to +\infty$.

Proof. For t such that

$$x_2(t) < (\alpha_2/\beta_2)\, x_1(t), \qquad \text{i.e., } \dot{x}_2(t) > 0, \tag{*}$$

we find $x_2(t)$ strictly increasing. Since $x_1(t)$ is decreasing to zero and $x_2(t) \geqslant 0$ we know that the inequality (*) cannot hold for all time. Let T_1 be the first time t with $t \geqslant r$ such that $\dot{x}_2(t) \leqslant 0$; then for all $\tau_2 > \tau_1 \geqslant T_1$ we shall show that $x_2(\tau_2) < x_2(\tau_1)$, i.e., $x_2(t)$ is monotonically decreasing.

Suppose that there exist τ_1 and τ_2 with $\tau_2 > \tau_1 \geqslant T_1$, such that $x_2(\tau_2) \geqslant x_2(\tau_1)$. In particular, we know that $x_2(t)$ is decreasing (at least nonincreasing) at T_1 so follow $x_2(t)$ until $x_2(t)$ is nondecreasing (this must occur if $x_2(\tau_2) \geqslant x_2(\tau_1)$). Let t^* be such that $x_2(t)$ is decreasing on $[T_1, t^*)$ and nondecreasing at least on $[t^*, \tau_2]$. We take $\tau_1 = t^*$ and then $x_2(\tau_1) \leqslant x_2(\tau_2)$. In the special case where $\dot{x}_2(T_1) = 0$ and $\dot{x}_2(t) \geqslant 0$ on some interval after T_1 we take $\tau_1 = T_1$. At $t = \tau_1$ we see that

$$\dot{x}_2(\tau_1) = \alpha_2 x_1(\tau_1) - \beta_2 x_2(\tau_1) \leqslant 0$$

while at τ_2 we have

$$\dot{x}_2(\tau_2) = \alpha_2 x_1(\tau_2) - \beta_2 x_2(\tau_2) \geqslant 0.$$

Therefore, $\alpha_2 x_1(\tau_2) - \beta_2 x_2(\tau_2) \geqslant \alpha_2 x_1(\tau_1) - \beta_2 x_2(\tau_1)$ which implies

$$\beta_2 x_2(\tau_2) \leqslant \beta_2 x_2(\tau_1) + \alpha_2(x_1(\tau_2) - x_1(\tau_1)). \tag{**}$$

But $x_1(t)$ is strictly decreasing which implies $x_1(\tau_2) < x_1(\tau_1)$. Using this fact together with (**) we find that $x_2(\tau_2) < x_2(\tau_1)$ which contradicts the assumption that there exist τ_1 and τ_2 such that $x_2(\tau_2) \geqslant x_2(\tau_1)$.

From this we see that after some finite time T_1, $\dot{x}_2(t) < 0$. Since $x_1(t) \searrow 0$, we see that $\dot{x}_2(t) + \beta_2 x_2(t) \to 0$ as $t \to +\infty$ which implies $x_2(t) \searrow 0$. So by a similar argument we obtain a finite time $T_2 \geqslant T_1$ such that $\dot{x}_3(t) < 0$ for all $t \geqslant T_2$ and $x_3(t) \searrow 0$. Continuing in the same manner we find there exist times $T_i \geqslant T_{i-1}$ such that $\dot{x}_{i+1}(t) < 0$ for all $t \geqslant T_i$ and $x_{i+1}(t) \searrow 0$. Set $T_0 = T_{n-1}$ and the lemma is established.

From Lemma 4.1 we see that if t_0 were infinite and if $z(t)$ remained in $S(1,\dots, 1)$ then after some finite time the trajectory must be arbitrarily close to the origin and all x_i's montonically decreasing.

LEMMA 4.2. *Assume* (H1) *and suppose* $0 \leqslant x_n \leqslant \delta$, *for all* $t \geqslant T$ *for some finite* T, *then* t_0 *is finite.*

Proof. Suppose t_0 is infinite. Take $T \geqslant T_0 + r$ where T_0 is as in Lemma 4.1 and such that $0 \leqslant x_n(t) \leqslant \delta$, $t \geqslant T$. Lemma 4.1 gives $x_n(t) \searrow 0$ so such a T exists. For $t \in [T, T+r]$ we first note that $f(x_n(t-r)) \leqslant 0$ as is easily seen from the formula for f and the fact that $x_n(t-r) \geqslant 0$. Thus (H1) implies $-cx_n(t) \geqslant f(x_n(t))$ using the assumption that $x_n(t) \geqslant 0$. Now using (4.2a), the monotonicity of $x_n(t)$, and only considering $t \in [T, T+r]$, we see that

$$\begin{aligned}
x_1(t) &= \int_T^t f(x_n(s-r))\, e^{-b(t-s)}\, ds + x_1(T)\, e^{-b(t-T)} \\
&\leqslant x_1(T)\, e^{-b(t-T)} - c \int_T^t x_n(s-r)\, e^{-b(t-s)}\, ds \\
&\leqslant x_1(T)\, e^{-b(t-T)} - cx_n(T) \int_T^t e^{-b(t-s)}\, ds \\
&= x_1(T)\, e^{-b(t-T)} - cx_n(T)\, e^{-bt}(e^{bt} - e^{bT})/b \\
&\leqslant x_1(T)\, e^{-bt}[e^{bT} - (c\alpha/b)(e^{bt} - e^{bT})].
\end{aligned}$$

Therefore,

$$x_1(T + r) \leqslant x_1(T)e^{-b(T+r)}[e^{bT} - (c\alpha/b)(e^{b(T+r)} - e^{bT})]$$
$$= x_1(T)e^{-br}[1 - (c\alpha/b)(e^{br} - 1)].$$

So by (H1), $x_1(T + r) < 0$ which contradicts t_0 being infinite. (From $x_1(T + r) < 0$ it would follow that there exists a $t_0 \in [T, T + r]$ such that $x_1(t_0) = 0$.)

Q.E.D.

Thus, we have demonstrated that in finite time t_0, $x_1(t_0) = 0$. In fact, we can show that the solution actually passes into $S(0, 1,..., 1)$.

LEMMA 4.3. *Assume* (H1). *Then the trajectory* $z(t)$ *enters* $S(0, 1,..., 1)$. *Furthermore,* $x_1(t) < 0$ *for all* t *such that* $x_n(t - r) \geqslant 0$ *and such that* $t \geqslant \tau$ *for some* τ *with* $z(\tau) \in S(0, 1,..., 1)$.

Proof. If at t_0, $x_n(t_0 - r) > 0$, then clearly $\dot{x}_1(t_0) < 0$ which implies the solution actually passes into $S(0, 1,..., 1)$. Now suppose that $x_n(t - r) = 0$ for $t \in [0, s]$ with $s \leqslant r$, then $\dot{x}_1(t) = -bx_1(t)$ or $x_1(t) = e^{-bt}x_1(0)$. Thus, we can easily see that $x_1(t) \geqslant 0$ on $[0, s]$ and $x_1(t) = 0$ on $[0, s]$ if and only if $x_1(0) = 0$. From this we can conclude that if $x_1(0) > 0$, then $t_0 > s$.

Since $e^{\beta_n t}x_n(t)$ is nondecreasing either $x_n(t) \equiv 0$ on $[-r, 0]$ or there exists an $s - r < 0$ such that $x_n(t) = 0$ on $[-r, s - r]$ and $x_n(t) > 0$ on $(s - r, 0]$. In the latter case we know $\dot{x}_n(t) \geqslant -\beta_n x_n(t)$ for $0 \leqslant t \leqslant t_0$ so we see that $x_n(t) \geqslant e^{-\beta_n t}x_n(0)$ for $0 \leqslant t \leqslant t_0$. This implies $x_n(t) > 0$ on $(s - r, t_0]$, which in turn implies that if $x_1(0) > 0$ then $x_n(t_0 - r) > 0$ and so the solution passes into $S(0, 1,..., 1)$.

Now consider the case when $x_1(0) = 0$ and $x_n(t) \not\equiv 0$ on $[-r, 0]$. In this case $f(x_n(t)) < 0$ for $t \in (s - r, 0]$. Using (4.2b) we see that for $t \in (s, r]$

$$x_1(t) = \int_0^t f(x_n(\tau - r))\, e^{-b(t-\tau)}\, d\tau$$
$$= \int_s^t f(x_n(\tau - r))\, e^{-b(t-\tau)}\, d\tau < 0,$$

which implies $z(t)$ passes into $S(0, 1,..., 1)$.

Next consider the possibility $x_n(t) \equiv 0$ on $[-r, 0]$. Since we are starting with initial data in $K_0\backslash\{0\}$ there is some i such that $x_i(0) > 0$. Let i_0 be the greatest i such that $x_i(0) > 0$. From repeated use of (4.2b) and using $x_i(0) = 0$ for $i > i_0$ we see that

$$x_n(t) = \int_0^t \alpha_n x_{n-1}(s_1)\, e^{-\beta_n(t-s_1)}\, ds_1 + x_n(0)\, e^{\beta_n t}$$
$$= \int_0^t \alpha_n e^{-\beta_n(t-s_1)} \left[\int_0^{s_1} \alpha_{n-1} e^{-\beta_{n-1}(s_1-s_2)} x_{n-2}(s_2)\, ds_2 + x_{n-1}(0)\, e^{\beta_{n-1}s_1}\right] ds_1$$

$$= \prod_{i=i_0+1}^{n} \alpha_i \int_0^t e^{-\beta_n(t-s_1)} \left[\int_0^{s_1} e^{-\beta_{n-1}(s_1-s_2)} \left[\cdots\right.\right.$$

$$\int_0^{s_{n-i_0-1}} \exp[-\beta_{i_0+1}(s_{n-i_0-1} - s_{n-i_0})]\, x_{i_0}(s_{n-i_0})\, ds_{n-i_0}$$

$$\left.\left. + x_{i_0+1}(0) \exp(-\beta_{i_0+1} s_{n-i_0-1})\right] ds_{n-i_0-1} \cdots\right] ds_1 .$$

But $x_{i_0}(0) > 0$ and $x_{i_0}(t)$ is continuous since it satisfies (2.2) so it can be seen from the above multiple integral that at least in some small neighborhood say $t < \epsilon$, $x_{i_0}(t) > 0$ and so $x_n(t) > 0$ for $0 < t < \epsilon$. In fact, for $t \leqslant t_0$ we remain in $S(1,..., 1)$ and so $x_n(t) > 0$ for $0 < t \leqslant t_0$.

Assuming $x_n(t) \equiv 0$ on $[-r, 0]$, then as before if $x_1(0) > 0$ then $x_1(r) > 0$. In this case $t_0 > r$ so $x_n(t_0 - r) > 0$ and we find that $z(t)$ enters $S(0, 1,..., 1)$. If $x_1(0) = 0$, then $x_1(t) = 0$ on $[0, r]$. Using (4.2a) we see that

$$x_1(t) = \int_r^t f(x_n(\tau - r))\, e^{-b(t-\tau)}\, d\tau + x_1(r)\, e^{-b(t-r)}$$

$$= \int_r^t f(x_n(\tau - r))\, e^{-b(t-\tau)}\, d\tau < 0 \qquad \text{for} \quad r < t < \epsilon,$$

so again $z(t)$ enters $S(0, 1,..., 1)$.

Since $z(t)$ enters $S(0, 1,..., 1)$ there exists $\tau > t_0$ such that $x_1(\tau) < 0$. From (2.2) we observe that whenever $x_n(t - r) \geqslant 0$, $t > \tau$, (equivalently $f(x_n(t - r)) \leqslant 0$), $\dot{x}_1(t) \leqslant -bx_1(t)$ and so $x_1(t) \leqslant x_1(\tau)e^{-b(t-\tau)} < 0$. So $z(t)$ passes into $S(0, 1,..., 1)$ and $x_1(t)$ remains negative at least until $x_n(t - r) < 0$. Q.E.D.

Now we have shown the solution $z(t)$ must enter $S(0, 1,..., 1)$ where from the equation for x_2 we have $\dot{x}_2 < 0$. From Lemma 4.3 we have $x_1(t) < 0$ since $x_n(t - r) \geqslant 0$ in $S(1,..., 1)$ and $S(0, 1,..., 1)$. As before, it is easily demonstrated that if at some time t_1, $x_i(t_1) = 0$ for some $i = 3, 4,..., n$, then $\dot{x}_i(t_1) \geqslant 0$ which implies the trajectory remains in $S(0, 1,..., 1)$. Since the trajectory can only exit through the hyperplane $x_2 = 0$, using the differential equation for x_2 we see that $x_2(t)$ is strictly decreasing to 0 in $S(0, 1,..., 1)$. Suppose $x_2(t_1) = 0$, for some t_1, $t_0 < t_1 \leqslant +\infty$.

LEMMA 4.4. *Assume* (H1); *then* t_1 *is finite.*

Proof. Suppose $t_1 = +\infty$. Since $z(t)$ is in $S(0, 1,..., 1)$ and since $x_2(t)$ is strictly decreasing to 0, in an analogous manner to proving Lemma 4.1 we can demonstrate that in some finite time T, $\dot{x}_i(t) < 0$ for all $t \geqslant T$, $i = 2,..., n$ and $x_i(t) \searrow 0$, $i \geqslant 2$. As before $\dot{x}_i(t) < 0$, $i \geqslant 2$, implies $x_n(t) > (\alpha_n/\beta_n)\, x_{n-1}(t) > \cdots > [\prod_{j=3}^{n} (\alpha_j/\beta_j)]\, x_2(t)$ for all $t \geqslant T$. Since t_1 is assumed to be infinite the above argument implies there exists a time $T_0 > T + r$, such that $x_n(t) \leqslant \delta$

for all $t \geqslant T_0 - r$ and δ as in (H1). Since c is fixed with $c > (b/\alpha)(e^{\beta_2 r} - 1)^{-1}$, we can choose $1 > \epsilon > 0$ satisfying

$$(b/(1-\epsilon)\alpha)(e^{\beta_2 r} - 1)^{-1} = c.$$

Now as before (H1) implies $-cx_n(t) \geqslant f(x_n(t))$ and since $x_1(t) < 0$, using (4.2a) we see that for $t \in [T_0, T_1 + r]$, where $T_1 \equiv T_0 + (1/b)\ln(1/\epsilon)$,

$$\begin{aligned} x_1(t) &= \int_{T_0}^{t} f(x_n(s-r))\, e^{-b(t-s)}\, ds + x_1(T_0)\, e^{-b(t-T_0)} \\ &\leqslant \int_{T_0}^{t} f(x_n(s-r))\, e^{-b(t-s)}\, ds \leqslant -\frac{c}{b}\, x_n(T_1)[1 - e^{-b(t-T_0)}]. \end{aligned}$$

Thus,

$$x_1(t) \leqslant -\frac{c(1-\epsilon)}{b}\, x_n(T_1), \qquad \text{for} \quad t \in [T_1, T_1 + r].$$

Now using (4.2b) we have for $t \in [T_1, T_1 + r]$

$$\begin{aligned} x_2(t) &= \int_{T_1}^{t} \alpha_2 x_1(s)\, e^{-\beta_2(t-s)}\, ds + x_2(T_1)\, e^{-\beta_2(t-T_1)} \\ &\leqslant -\frac{c(1-\epsilon)}{b}\, \alpha_2 x_n(T_1) \int_{T_1}^{t} e^{-\beta_2(t-s)}\, ds + x_2(T_1)\, e^{-\beta_2(t-T_1)} \\ &\leqslant -\frac{c(1-\epsilon)}{b}\, \frac{\alpha_2}{\beta_2}\, x_n(T_1)[1 - e^{\beta_2(T_1-t)}] + x_2(T_1)\, e^{-\beta_2(t-T_1)} \\ &\leqslant -\frac{c(1-\epsilon)}{b}\, \alpha x_2(T_1)[1 - e^{\beta_2(T_1-t)}] + x_2(T_1)\, e^{-\beta_2(t-T_1)} \\ &= x_2(T_1)\left[-\frac{c(1-\epsilon)}{b}\, \alpha[1 - e^{\beta_2(T_1-t)}] + e^{-\beta_2(t-T_1)}\right]. \end{aligned}$$

In particular,

$$x_2(T_1 + r) \leqslant x_2(T_1)\left[-\frac{c(1-\epsilon)}{b}\, \alpha(1 - e^{-\beta_2 r}) + e^{-\beta_2 r}\right] = 0$$

by the choice of ϵ. But this implies there exists a $t_1 \in [T_1, T_1 + r]$ such that $x_2(t_1) = 0$ which contradicts $t_1 = +\infty$. Q.E.D.

Now by Lemma 4.4 we have t_1 finite, and since $x_1(t_1) < 0$ we see that $\dot{x}_2(t_1) = \alpha_2 x_1(t_1) < 0$ which implies $z(t)$ enters $S(0, 0, 1,..., 1)$. In a manner similar to the arguments in Lemma 4.3 we see that for all t such that $x_1(t) \leqslant 0$, $\dot{x}_2(t) \leqslant -\beta_2 x_2(t)$. So for some $\tau > t_1$ with $x_2(\tau) < 0$ we have $x_2(t) \leqslant x_2(\tau)e^{-\beta_2(t-\tau)} < 0$ for $t > \tau$ and t such that $x_1(t) \leqslant 0$.

This process continues in a similar manner so that at the ith stage we can similarly show that the trajectory can only exit through $x_i = 0$ and that $x_i(t)$ is strictly decreasing to 0. Let $x_i(t_{i-1}) = 0$.

LEMMA 4.5. *Assume* (H1). *Then* t_{i-1} *is finite.*

We shall only sketch how the proof of this would go as it is similar in nature to that of the previous lemma. Again, if we assume $t_{i-1} = +\infty$, then in some finite time T we can show that $\dot{x}_j(t) < 0$, $j \geqslant i$ for all $t \geqslant T$ so $x_n(t) > (\alpha_n/\beta_n)\, x_{n-1}(t) > \cdots > (\prod_{j=i+1}^{n} (\alpha_j/\beta_j))\, x_i(t)$ for all $t \geqslant T$ and that $x_j(t) \searrow 0$, $j \geqslant i$. Using this we have $x_n(t) \leqslant \delta$ for all $t \geqslant T_0 - r$, for some $T_0 \geqslant T + r$, where δ is as in (H1).

As before c is fixed in (H1) so there exists $1 > \epsilon > 0$, satisfying $[b/((1-\epsilon)^{i-1}\alpha)](e^{\beta_i r} - 1)^{-1} = c > (b/\alpha)(e^{\beta_i r} - 1)^{-1}$. Define $T_1 = T_0 + (1/b)\ln(1/\epsilon)$ and for each $j = 2,\ldots, i-1$, $T_j \equiv T_{j-1} + (1/\beta_j)\ln(1/\epsilon)$. Then again by using (4.2a) and assuming $t \in [T_0, T_{i-1} + r]$ we can show

$$x_1(t) \leqslant -\frac{c(1-\epsilon)}{b}\, x_n(T_{i-1}), \qquad \text{for} \quad t \geqslant T_0 + \frac{1}{b}\ln\frac{1}{\epsilon} = T_1 .$$

Then using (4.2b) we see that

$$\begin{aligned} x_2(t) &\leqslant -\frac{c(1-\epsilon)}{b}(\alpha_2/\beta_2)\, x_n(T_{i-1})(1 - e^{-\beta_2(t-T_1)}) \\ &\leqslant -\frac{c(1-\epsilon)^2}{b}(\alpha_2/\beta_2)\, x_n(T_{i-1}), \qquad \text{for} \quad t \geqslant T_2 . \end{aligned}$$

Continuing in a similar manner, we obtain for $j = 2, 3,\ldots, i-1$

$$x_j(t) \leqslant -\frac{c(1-\epsilon)^j}{b}\left[\prod_{k=2}^{j}(\alpha_k/\beta_k)\right] x_n(T_{i-1}), \qquad \text{for} \quad t \geqslant T_j .$$

We can use this to argue that for $t \in [T_{i-1}, T_{i-1} + r]$

$$\begin{aligned} x_i(t) &= \int_{T_{i-1}}^{t} \alpha_i x_{i-1}(s)\, e^{-\beta_i(t-s)}\, ds + x_i(T_{i-1})\, e^{-\beta_i(t-T_{i-1})} \\ &\leqslant -\frac{c(1-\epsilon)^{i-1}}{b}\,\alpha_i\left[\prod_{j=2}^{i-1}(\alpha_j/\beta_j)\right] x_n(T_{i-1})[(1 - e^{-\beta_i(t-T_{i-1})})/\beta_i] \\ &\quad + x_i(T_{i-1})\, e^{-\beta_i(t-T_{i-1})} \\ &\leqslant -\frac{c(1-\epsilon)^{i-1}}{b}\,\alpha x_i(T_{i-1})[1 - e^{-\beta_i(t-T_{i-1})}] + x_i(T_{i-1})\, e^{-\beta_i(t-T_{i-1})} \end{aligned}$$

Therefore,

$$x_i(T_{i-1}+r) \leqslant x_i(T_{i-1})\, e^{-\beta_i r}\left[-\frac{c(1-\epsilon)^{i-1}}{b}\,\alpha(e^{\beta_i r}-1)+1\right]=0,$$

which implies there exists $t_{i-1} \in [T_{i-1}, T_{i-1}+r]$ such that $x_i(t_{i-1}) = 0$ contradicting the assumption that t_{i-1} is infinite. Q.E.D.

Lemma 4.5 shows that $x_i(t_{i-1}) = 0$ in finite time, so we see that $\dot{x}_i(t_{i-1}) = \alpha_i x_{i-1}(t_{i-1}) < 0$ and thus $z(t)$ enters

$$S(\overbrace{0,\ldots,0}^{i}, 1,\ldots, 1).$$

As before we can show that for each $i \geqslant 2$, $x_i(t) < 0$ for all t with $t > t_{i-1}$ and such that $x_{i-1}(t) \leqslant 0$. Using the above lemmas and this information we see that in finite time t_{n-1} the trajectory $z(t)$ crosses $x_n = 0$ and then enters $S(0,\ldots, 0)$. In fact, it is easy to trace its path and see that it cycles through the orthants as described previously.

Suppose $z(t) \in S(0,\ldots, 0)$. If we assume the trajectory first exits through any face $x_i = 0$, for some $i \geqslant 2$, say $x_i(t_n) = 0$, then (since $x_{i-1}(t_n) \leqslant 0$) by the comment after Lemma 4.5, $x_i(t_n) < 0$ which is a contradiction. So if the trajectory exits it must exit through $x_1 = 0$.

For $t_{n-1} \leqslant t \leqslant t_{n-1} + r$, $x_n(t-r) \geqslant 0$ which implies by Lemma 4.3 $x_1(t) < 0$. Since $z(t)$ can only exit $S(0,\ldots, 0)$ through $x_1 = 0$, this implies that $z(t)$ remains at least r units of time in $S(0,\ldots, 0) \cup \{x_n = 0\}$.

We must still show that $-z_{t_{n-1}+r}(\psi_1) \in K\backslash\{0\}$. Clearly $z_{t_{n-1}+r}(\psi_1) \neq 0$ and so we only need to show $e^{\beta_n t}x_n(t)$ is nonincreasing on $[t_{n-1}, t_{n-1}+r]$. From the above comments we know that $z(t) \in S(0,\ldots, 0) \cup \{x_n = 0\}$ on this interval. Using (4.2b) we obtain

$$\begin{aligned} e^{\beta_n t}x_n(t) &= \alpha_n \int_{t_{n-1}}^{t} x_{n-1}(s)\, e^{\beta_n s}\, ds + x_n(t_{n-1})\, e^{\beta_n t_{n-1}} \\ &= \alpha_n \int_{t_{n-1}}^{t} x_{n-1}(s)\, e^{\beta_n s}\, ds \qquad \text{for} \quad t \in [t_{n-1}, t_{n-1}+r], \end{aligned}$$

which is clearly nonincreasing since the integrand $x_{n-1}(s) \leqslant 0$ on $[t_{n-1}, t_{n-1}+r]$ (in fact, $x_{n-1}(s) < 0$). Therefore, we have shown that given $\psi_1 \in K_0\backslash\{0\}$, that $\tau_1(\psi_1) = t_{n-1} + r$ and $z_{\tau_1(\psi_1)}(\psi_1) \in K_1\backslash\{0\}$.

It remains to show that $\tau_1(\psi)$ is continuous in ψ. We shall do this by considering the nth component of the trajectory at t_{n-1}. Above we have shown that given any $\psi \in K_0\backslash\{0\}$, there exists $t_{n-1}(\psi)$ such that $x_n(t_{n-1}(\psi), \psi) = 0$ and $\dot{x}_n(t_{n-1}) < 0$. By our choice of K_0 and K_1 there exists an open set $\mathscr{U} \subseteq \mathscr{C}_0$ containing K_0 and K_1 such that our existence theorem (Mahaffy [17]) gives us a unique solution for each $\psi \in \mathscr{U}$. From Hale [10, p. 41], we have $x_n(t, \psi)$ depends continuously on both t and ψ, i.e., $z(t)$ exists and is continuous on $[0, +\infty)$ so if we let $z^k(t)$ denote the solution at time t with initial data ψ_k and if $\{\psi_k\} \to \psi_0$ in the $\mathscr{C}_0$ topology, then for any $b < +\infty$ there exists a $k_0 = k_0(b)$ such that for $k \geq k_0$ we have $z^k(t) \to z^0(t)$ uniformly on $[0, b]$ and so $x_n(t, \psi_k) \to x_n(t, \psi_0)$. We want to show that $t_{n-1}(\psi)$ is continuous in ψ.

Fix $\psi_1 \in K_0\backslash\{0\}$. Then $x_n(t_{n-1}(\psi_1); \psi_1) = 0$ and $\dot{x}_n(t_{n-1}) < 0$, so by the Implicit Function Theorem (Lang [15, p. 125]) there exist neighborhoods $\mathscr{V}_1$ and $\mathscr{U}_1$ of t_{n-1} and ψ_1 and a locally unique continuous function $\tau_1: \mathscr{U}_1 \to \mathscr{V}_1 + r$ such that $x_n(\tau_1(\psi) - r; \psi) = 0$ for all $\psi \in \mathscr{U}_1$. By uniqueness $\tau_1(\psi) - r = t_{n-1}(\psi)$ on $\mathscr{U}_1$ and so $t_{n-1}(\psi)$ is continuous in ψ locally. However, we can do this for any $\psi_1 \in K_0\backslash\{0\}$, so we can extend this to all of $K_0\backslash\{0\}$. Thus we have found our continuous map $\tau_1: K_0\backslash\{0\} \to (r, \infty)$ with $\tau_1(\psi) = t_{n-1}(\psi) + r$ and $z_{\tau_1(\psi)}(\psi) \in K_1\backslash\{0\}$.

As stated at the beginning of the proof, a very similar proof gives (ii). Observe that in part (iii) $s_1 = t_0$ and $s_2 = t_n$ where t_n is the time when $x_1(t_n) = 0$ and the trajectory is passing into $S(1, 0,..., 0)$. Then since (ii) implies we have initial conditions for (i) again the cycle repeats and so on giving us (iii), and thus completing the proof of the theorem. Q.E.D.

5. Periodic Solutions of the Goodwin-Type Models

Theorem 4.1 gives us the oscillatory phenomenon of the system (2.2). To show periodicity we want to make use of Theorem 2.1. We define the operator $\mathscr{T}: K_0 \to K_0$ by

$$\mathscr{T}\psi = z_{\tau(\psi)}(\psi)$$

where $\tau(\psi) = \tau_1(\psi) + \tau_2(z_{\tau_1(\psi)}(\psi))$ and $z_{\tau(\psi)}(\psi) \in K$ with $z_t(\psi)$ a solution of (2.2). We must show that the map $\mathscr{T}$ is completely continuous and τ is bounded continuous.

Proposition 5.1. *The map $\mathscr{T}$: $K_0\backslash\{0\} \to K_0\backslash\{0\}$ is completely continuous.*

Proof. Since $z_t(\psi)$ is continuous in t and ψ, and $\tau(\psi)$ is continuous in ψ as shown above, $\mathscr{T}\psi = z_{\tau(\psi)}(\psi)$ is continuous in ψ. Let $\{\mathscr{T}\psi_k\}_{k=1}^{\infty}$ be any sequence in K_0. Since K_0 is bounded in the $\mathscr{C}_0$ topology, $\{\mathscr{T}\psi_k\}_{k=1}^{\infty}$ is uniformly bounded by the bound on K_0 which implies $\{\mathscr{T}\psi_k\}_{k=1}^{\infty}$ is equibounded. Since $z_t(\psi_k)$ is a solution of (2.2) and using the fact that $z(t) \in \mathscr{R}$ we can use (2.2) to obtain a uniform bound on the derivatives of $z_t(\psi_k)$. In particular,

$$| x_n(s_2) - x_n(s_1)| \leq \int_{s_1}^{s_2} | \dot{x}_n(\sigma)| \, d\sigma \leq M' \, | s_2 - s_1 |$$

for some M' independent of s_1, s_2, and ψ_k. This implies $x_n^{\tau(\psi_k)}$ is an equicontinuous family of functions. (Recall $x_n{}^t \in \mathscr{C}([-r, 0]; R)$ stands for the function $x_n(t+\theta)$ for $\theta \in [-r, 0]$.) Using the Ascoli–Arzelà Theorem (Yosida [20, p. 85]) we have that the sequence $\{x_n^{\tau(\psi_k)}\}_{k=1}^{\infty}$ has a convergent subsequence. The other components $x_i(\tau(\psi_k)) \in R$, $i = 1,..., n-1$ form a sequence which have a convergent subsequence by the equiboundedness. Combining these facts we have $\{\mathscr{T}\psi_k\}_{k=1}^{\infty}$ is relatively compact in K_0 which implies $\mathscr{T}: K_0\backslash\{0\} \to K_0\backslash\{0\}$ is completely continuous. Q.E.D.

Now we must obtain a uniform bound on the map τ.

PROPOSITION 5.2. *The map* τ: $K_0\backslash\{0\} \to [r, \infty)$ *is bounded continuous.*

Proof. We have shown τ is continuous in ψ, so it suffices to show that the map τ is equibounded. In Section 4, we showed that for each $\psi \in K_0\backslash\{0\}$, $\tau(\psi)$ is finite. However, in this proposition we need to extend those results to obtain a uniform bound independent of the initial data. Let t_i, $i = 1,..., n-1$, and τ_1 be as before, then we shall show that there exists a constant M such that $\tau_1(\psi) \leqslant M$ for all $\psi \in K_0\backslash\{0\}$. A similar argument will bound τ_2 and thus we obtain our uniform bound on the map τ.

For clarity we shall divide the argument into two lemmas. The first will bound the time for which the nth component of z can be outside a neighborhood of the origin. The second will bound the time t_0 independent of initial data.

LEMMA 5.1. *Assume* $\tau_1(\cdot)$ *is sufficiently large. Then there exists a time* $S_n = S_n(\delta)$ *such that* $x_n(t) \leqslant \delta$ *for all* $S_n \leqslant t \leqslant \tau_1$ *where* δ *is as in* (H1).

Proof. Let $\delta_n = \delta$ where δ is as in (H1) and then choose δ_{i-1} such that $\delta_{i-1} = \beta_i\delta_i/2\alpha_i$, $i = 2,..., n$. Define $N = \max_{x\in\mathscr{R}} \max_{1\leqslant i\leqslant n}\{|x_i|\}$. We have already shown that for $0 \leqslant t \leqslant \tau_1$, $x_n(t-r) \geqslant 0$, which implies $f(x_n(t-r)) \leqslant 0$. Using this fact we see that $\dot{x}_1(t) \leqslant -bx_1(t)$ for $0 \leqslant t \leqslant \tau_1$. From the basic differential inequalities (Coppel [6, p. 28]) we find that

$$x_1(t) \leqslant e^{-bt}\, x_1(0) \leqslant e^{-bt}N \qquad \text{for} \quad 0 \leqslant t \leqslant \tau_1 .$$

So for τ_1 sufficiently large we see there exists $S_1 = S_1(\tau_1)$ such that $x_1(t) \leqslant \delta_1$ for all $S_1 \leqslant t \leqslant \tau_1$ and independent of initial data.

Now assume $S_1 \leqslant t \leqslant \tau_1$ then $\dot{x}_2(t) \leqslant \alpha_2\delta_1 - \beta_2 x_2(t)$. Again, we can use differential inequalities to obtain

$$\begin{aligned} x_2(t) &\leqslant x_2(S_1)\, e^{-\beta_2(t-S_1)} + \int_{S_1}^{t} \alpha_2\delta_1 e^{-\beta_2(t-s)}\, ds \\ &= x_2(S_1)\, e^{-\beta_2(t-S_1)} + (\alpha_2\delta_1/\beta_2)(1 - e^{-\beta_2(t-S_1)}) \\ &\leqslant e^{-\beta_2(t-S_1)}N + (\alpha_2\delta_1/\beta_2). \end{aligned}$$

As before, if τ_1 is sufficiently large then there exists $S_2 = S_2(\delta_2)$ such that $x_2(t) < \delta_2$ for $S_2 \leqslant t \leqslant \tau_1$ where $\delta_2 \equiv (2\alpha_2\,\delta_1/\beta_2)$.

Continuing in the same manner we see that for $S_{i-1} \leqslant t \leqslant \tau_1$

$$x_i(t) < e^{-\beta_i(t-S_{i-1})}N + (\alpha_i\delta_{i-1}/\beta_i),$$

so for τ_1 sufficiently large there exists $S_i = S_i(\delta_i)$ such that $x_i(t) < \delta_i$ for $S_i \leqslant t \leqslant \tau_1$. In particular, there exists $S_n = S_n(\delta_n)$ such that $x_n(t) \leqslant \delta_n$ for all $S_n \leqslant t \leqslant \tau_1$, and this is independent of the initial data. Q.E.D.

We now want to demonstrate the following:

LEMMA 5.2. *The time t_0 of Section 4 is bounded independent of initial data.*

Proof. We can assume without loss of generality that t_0 can be made arbitrarily large for an appropriate choice of ψ. If not we would have our uniform bound and the proof would be complete. For $0 \leqslant t < t_0$ we have $x_1(t) > 0$. From Eq. (4.2a) we see that if $f_1(s) = f(x_n(s-r))$,

$$x_1(t) = e^{-bt}\left[x_1(0) + \int_0^t f_1(s)\, e^{bs}\, ds\right].$$

For $t < t_0$ using (4.2b) we see that

$$\begin{aligned} x_2(t) &= e^{-\beta_2 t}\left[x_2(0) + \int_0^t \alpha_2 x_1(s)\, e^{\beta_2 s}\, ds\right] \\ &\geqslant e^{-\beta_2 t}\left[\alpha_2 \int_0^t \left[x_1(0) + \int_0^s f_1(\tau)\, e^{b\tau}\, d\tau\right] e^{(\beta_2 - b)s}\, ds\right] \\ &= e^{-\beta_2 t}\left[\alpha_2 x_1(0) \int_0^t e^{(\beta_2-b)s}\, ds + \alpha_2 \int_0^t f_1(\tau)\, e^{b\tau} \int_\tau^t e^{(\beta_2-b)s}\, ds\, d\tau\right] \end{aligned}$$

since f_1 is continuous and bounded. Let us assume $b \neq \beta_2$ to proceed with our calculations. The special case $b = \beta_2$ is handled in the same manner.

$$\begin{aligned} x_2(t) \geqslant e^{-\beta_2 t}\Big[&\alpha_2 x_1(0)((e^{(\beta_2-b)t} - 1)/(\beta_2 - b)) \\ &+ \alpha_2 \int_0^t f_1(\tau)\, e^{b\tau}((e^{(\beta_2-b)t} - e^{(\beta_2-b)\tau})/(\beta_2 - b))\, d\tau\Big]. \end{aligned}$$

But for $t < t_0$, $x_n(t-r) \geqslant 0$ which implies $f_1(t) \leqslant 0$ and so

$$x_2(t) \geqslant e^{-\beta_2 t}[(e^{(\beta_2-b)t} - 1)/(\beta_2 - b)]\left[\alpha_2\left(x_1(0) + \int_0^t f_1(\tau)\, e^{b\tau}\, d\tau\right)\right].$$

Assuming $t_0 > r$, we have

$$x_2(r) \geqslant e^{-\beta_2 r}[(e^{(\beta_2-b)r} - 1)/(\beta_2 - b)] \left[\alpha_2 \left(x_1(0) + \int_0^r f_1(\tau)\, e^{b\tau}\, d\tau\right)\right]$$

$$\equiv A_1 e^{-br} \left[\alpha_2 \left(x_1(0) + \int_0^r f_1(\tau)\, e^{b\tau}\, d\tau\right)\right]$$

$$= A_1 \alpha_2 x_1(r) \qquad \text{with } A_1 \text{ independent of initial conditions.}$$

As in the proof of Lemma 4.1 we want to find a time T_1 such that for all $t \geqslant T_1$, $x_2(t)$ is monotonically decreasing. Take $T_1 \geqslant r$ such that $e^{-b(T_1-r)} < A_1\beta_2$ and assume without loss of generality that $T_1 \leqslant t_0$. Recall from Lemma 4.1 that we showed that once $\dot{x}_2(t') < 0$ for some t', then $\dot{x}_2(t) < 0$ for at least $t' \leqslant t \leqslant t_0$. Suppose for the moment that x_2 is nondecreasing at least until $t = T_1$ then $x_2(T_1) \geqslant x_2(r)$. But from (4.2a)

$$x_1(T_1) = e^{-b(T_1-r)} \left[x_1(r) + \int_r^{T_1} f_1(s)\, e^{bs}\, ds\right] \leqslant e^{-b(T_1-r)} x_1(r).$$

Using this we see that

$$x_2(T_1) \geqslant x_2(r) \geqslant A_1\alpha_2 x_1(r) \geqslant A_1 \alpha_2 e^{b(T_1-r)} x_1(T_1) > (\alpha_2/\beta_2)\, x_1(T_1)$$

by choice of T_1, which implies $\dot{x}_2(T_1) < 0$ contradicting x_2 nondecreasing until $t = T_1$. Note that T_1 is independent of the initial conditions ψ.

Now we shall show $\dot{x}_3(t) < 0$ for $t \geqslant T_2$ for some T_2 independent of the initial conditions (assuming t_0 sufficiently large). Using (4.2b) we see

$$x_3(T_1 + 1) = e^{-\beta_3(T_1+1)} \left[x_3(T_1)\, e^{\beta_3 T_1} + \alpha_3 \int_{T_1}^{T_1+1} x_2(s)\, e^{\beta_3 s}\, ds\right]$$

$$\geqslant (\alpha_3/\beta_3)(1 - e^{-\beta_3})\, x_2(T_1 + 1).$$

From (4.2b) and knowing $x_1(t) \leqslant x_1(t')e^{-b(t-t')}$ we find that

$$x_2(t) = e^{-\beta_2[t-(T_1+1)]} \left[x_2(T_1 + 1) + \int_{T_1+1}^{t} \alpha_2 x_1(s)\, e^{\beta_2 s}\, ds\right],$$

$$x_2(t) \leqslant e^{-\beta_2[t-(T_1+1)]} \left[x_2(T_1 + 1) + \alpha_2 x_1(T_1 + 1) \int_{T_1+1}^{t} e^{-b[s-(T_1+1)]} e^{\beta_2 s}\, ds\right],$$

$$x_2(t) < e^{-\beta_2[t-(T_1+1)]} x_2(T_1 + 1) \left[1 + \beta_2 e^{b(T_1+1)} \int_{T_1+1}^{t} e^{(\beta_2-b)s}\, ds\right]. \tag{5.1}$$

We define a function $\zeta(t)$ in the following manner: If $b \neq \beta_2$

$$\zeta(t) \equiv [c_1 e^{-\beta_2 t} + c_2 e^{-\beta_2 t}[(e^{(\beta_2-b)[t-(T_1+1)]} - 1)/(\beta_2 - b)]],$$

where $c_1 = e^{\beta_2(T_1+1)}$ and $c_2 = \beta_2 e^{2b(T_1+1)}$, and if $b = \beta_2$

$$\zeta(t) \equiv [c_1 + c_3(t - (T_1 + 1))]\, e^{-\beta_2 t},$$

where c_1 as before and $c_3 = \beta_2 e^{\beta_2(T_1+1)}$. Using (5.1) we see that $x_2(t) < \zeta(t)\, x_2(T_1 + 1)$. However, one can easily see from the definition of ζ that $\zeta(t)$ is an exponentially decreasing function of t, and so there exists a time $T_2 > T_1 + 1$ such that $\zeta(T_2) \leqslant (1 - e^{-\beta_3})$. Clearly, T_2 depends only on b, β_2, β_3, and T_1. If we suppose x_3 is nondecreasing at least until $t = T_2$, then

$$x_3(T_2) \geqslant x_3(T_1 + 1) \geqslant (\alpha_3/\beta_3)(1 - e^{-\beta_3})\, x_2(T_1 + 1) > (\alpha_3/\beta_3)\, x_2(T_2)$$

by our choice of T_2, which is a contradiction of $x_3(t)$ nondecreasing until $t = T_2$.

Assuming we do not reach t_0, this argument can be continued in a similar fashion until we reach T_{n-1} which depends only upon the parameters of the system and not ψ the initial data. From this we have for $t > T_{n-1}$, $x_i(t)$ decreasing for $i = 1,..., n$, and so

$$x_n(t) > (\alpha_n/\beta_n)\, x_{n-1}(t) > \cdots > \prod_{i=2}^{n} (\alpha_i/\beta_i)\, x_1(t).$$

Now take $T = \max\{T_{n-1} + r,\ S_n\}$, where S_n is as in Lemma 5.1. Then arguments such as those in the proof of Lemma 4.2 reveal that $t_0 < T + r$ and so t_0 is bounded independent of initial data $\psi \in K_0\backslash\{0\}$. Q.E.D.

As in the above lemma for each t_{i-1} we can show that $\dot{x}_j(t) < 0$, $j \geqslant i$ for all t such that $T \leqslant t < t_{i-1}$ for some T depending only upon the parameters of the system (assuming t_{i-1} sufficiently large). Having found this T let $T_0 = \max\{T + r, S_n\}$, where S_n is as in Lemma 5.1. Then arguments similar to those behind Lemma 4.5 show that $t_{i-1} \leqslant T_{i-1} + r$ where T_{i-1} is as defined in that lemma and T_{i-1} is independent of the initial data. In particular we have a uniform bound on $t_{n-1} = \tau_1 - r$ which proves the proposition. Q.E.D.

It remains to prove that $0 \in \mathscr{C}_0$ is an ejective point of the map $\mathscr{T}$. (Recall our completely continuous map $\mathscr{T}$ is in the space $\mathscr{C}_0$, not $\mathscr{C}$.) We must establish an analog to Theorem 2.2 in order to show ejectivity of the origin in $\mathscr{C}_0$. The main problem is that in condition (ii) of Theorem 2.2 the projection π_λ is defined in $\mathscr{C}$ while we are interested in results in $\mathscr{C}_0$. We shall, therefore, analyze carefully π_λ as defined and employed in Hale [10].

Let λ_0 be a solution of the characteristic equation (2.6) with $\operatorname{Re} \lambda_0 > 0$. Let Ψ_{λ_0} be a basis for P_{λ_0}, where P_{λ_0} is the eigenspace of (2.3) associated with λ_0 in the decomposition of $\mathscr{C}$ by λ_0. If $P^*_{\lambda_0}$ is the eigenspace of the adjoint equation (2.5) associated with λ_0 then let Φ_{λ_0} be a basis for $P^*_{\lambda_0}$ with $(\Phi_{\lambda_0}, \Psi_{\lambda_0}) = I$, where (ϕ, ψ) is the associated bilinear form

$$(\phi, \psi) = \phi(0)\,\psi(0) + \int_{-r}^{0} \phi(\theta + r)\, B\psi(\theta)\, d\theta.$$

From Hale [10, p. 173 ff], we see that for any $\psi \in \mathscr{C}$, $\pi_{\lambda_0}\psi = \Psi_{\lambda_0}(\Phi_{\lambda_0}, \psi)$. From this we see that condition (ii) $\inf\{\| \pi_{\lambda_0}\psi \|: \psi \in \bar{K}, \| \psi \| = \delta\} > 0$ is equivalent to $\inf\{|(\Phi_{\lambda_0}, \psi)|: \psi \in \bar{K}, \| \psi \| = \delta\} > 0$. To analyze (Φ_{λ_0}, ψ) we must have a basis Φ_{λ_0} for the solutions of the adjoint equation (2.5). Let $\Phi_{\lambda_0}(t)$ be a $d \times n$ matrix function and $\psi(t) = (\xi_1(t),..., \xi_n(t))^T$ be an n-vector function, then using the bilinear form as in Hale [10, p. 178], we have the following:

$$(\Phi_{\lambda_0}, \psi) = \Phi_{\lambda_0}(0)\,\psi(0) + \int_{-r}^{0} \Phi_{\lambda_0}(\theta + r)(f'(0)\,\psi_n(\theta), 0,..., 0)^T\, d\theta, \qquad (5.2)$$

since the matrix B has only one nonzero element. If we fix our basis Φ_{λ_0}, then $(\Phi_{\lambda_0}, \psi) = b(\psi)$ with $b(\cdot)$ being a d-vector. From the special form of (5.2) one sees that (Φ_{λ_0}, ψ) can be thought of as a mapping defined on $R^{n-1} \times \mathscr{C}([-r, 0]; R)$ or $\mathscr{C}_0$, which is the space in which we are interested.

In the special case when λ_0 is simple then a basis Φ_{λ_0} of (2.5) has the form $\Phi_{\lambda_0}(s) = e^{-\lambda_0 s}\mathbf{b}$ for some eigenvector $\mathbf{b}$. Let $\mathbf{b} = (b_1,..., b_n)$ and $\psi(t) = (\xi_1(t),..., \xi_n(t))^T$, then (Φ_{λ_0}, ψ) becomes

$$(\Phi_{\lambda_0}, \psi) = \sum_{i=1}^{n} b_i\xi_i(0) + \int_{-r}^{0} b_1 f'(0)\, \xi_n(\theta)\, e^{-\lambda_0(\theta+r)}\, d\theta. \qquad (5.2a)$$

Define $(\Phi_{\lambda_0}, \cdot)_{\mathscr{C}_0}$ on $\mathscr{C}_0$ such that if we have ψ in $\mathscr{C}$ it agrees with (Φ_{λ_0}, ψ), that is, for $\psi \in \mathscr{C}$ as given above and for $\psi_0 \in \mathscr{C}_0$ with $\psi_0 = (\xi_1(0),..., \xi_{n-1}(0), \xi_n(t))^T$, $t \in [-r, 0]$, we let $(\Phi_{\lambda_0}, \psi_0)_{\mathscr{C}_0} = (\Phi_{\lambda_0}, \psi)$. We shall now show in a manner similar to that used in establishing Theorem 2.2 (see Hale [10, p. 250]) that for $\psi_0 = (\xi_1(0),..., \xi_{n-1}(0), \xi_n)^T \in \mathscr{C}_0$ we have the following:

Proposition 5.3. *Suppose the following conditions are fulfilled*:

(i) *There is a characteristic root* λ *of* (2.3) *satisfying* $\operatorname{Re}\lambda > 0$.

(ii) *There is a convex set* $K_0 \subseteq \mathscr{C}_0$, $0 \in K_0$, *and* $\delta > 0$, *such that*

$$v = v(\delta) \equiv \inf\{|(\Phi_{\lambda_0}, \psi_0)_{\mathscr{C}_0}|: \psi_0 \in K_0, \| \psi_0 \|_{\mathscr{C}_0} = \delta\} > 0.$$

(iii) *There is a bounded continuous function* $\tau: K_0\backslash\{0\} \to [r, \infty)$, *such that the function defined by*

$$\mathscr{T}\psi_0 = z_{\tau(\psi_0)}(\psi_0), \qquad \psi_0 \in K_0\backslash\{0\}$$

(*where* z_t *is a solution of* (2.2)), *maps* $K_0\backslash\{0\}$ *into* K_0 *and is completely continuous*.

Then $0 \in \mathscr{C}_0$ *is an ejective point of* $\mathscr{T}$.

Proof. We want to make use of as much of the machinery developed by Hale [10] as we can in $\mathscr{C}$ before projecting into $\mathscr{C}_0$ to establish this result. Let $\mathscr{P}: \mathscr{C} \to \mathscr{C}_0$ be the natural projection, that is, if $\psi_c = (\psi_1, \psi_2, \ldots, \psi_n)^T \in \mathscr{C}$, then

$$\mathscr{P}(\psi_c) = \psi = (\psi_1(0), \psi_2(0), \ldots, \psi_{n-1}(0), \psi_n)^T \in \mathscr{C}_0 .$$

We shall follow the proof of Theorem 2.3 in Hale [10, p. 251]. Let λ_0 be a solution of (2.6) with $\operatorname{Re} \lambda_0 > 0$. As stated before there exists a basis Ψ_{λ_0} for P_{λ_0} such that for any $\psi_c \in \mathscr{C}$, $\pi_{\lambda_0}\psi_c = \Psi_{\lambda_0} b$ with $b = b(\psi_c) = (\Phi_{\lambda_0}, \psi_c)$ where Φ_{λ_0} is a basis for $P^*_{\lambda_0}$. We want a positive definite quadratic functional V: $\mathscr{C}_0 \to R$ with $V(\psi) = \hat{b}^T D \hat{b}$ and with the property that, for any $p > 0$, there is a $\delta_0 > 0$ such that, for any δ, $0 \leqslant \delta \leqslant \delta_0$, $\dot{V}^*(\psi) \equiv \liminf_{t \to 0^+} (1/t)[V(z_t(\psi)) - V(\psi)] > 0$ if $V(\psi) \geqslant p^2 \delta^2$, $\psi \in \bar{B}_\delta \subseteq \mathscr{C}_0$. Here $\bar{B}_\delta$ is the closed ball of radius δ in $\mathscr{C}_0$. This will be used to show solutions starting in a neighborhood of the origin of $\mathscr{C}_0$ are forced out in finite time giving ejectivity in our space $\mathscr{C}_0$.

If we have a real basis then $b \in R^d$, where d is the dimension of P_{λ_0}, so $\mathscr{P}(\pi_{\lambda_0}\psi) = \mathscr{P}(\Psi_{\lambda_0} b) = \mathscr{P}(\Psi_{\lambda_0}) b \in \mathscr{C}_0$. Using (5.2) we see that for any $\psi_1, \psi_2 \in \mathscr{C}$ with $\mathscr{P}\psi_1 = \mathscr{P}\psi_2$, then $b(\psi_1) = b(\psi_2)$. Let $\psi \in \mathscr{C}_0$; then we can associate a vector $\hat{b}(\psi) \equiv (\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}$ so that whenever $\psi = \mathscr{P}\psi_c$ we have $b(\psi_c) = \hat{b}(\psi)$.

Following Hale [10, p. 232], if $\pi_{\lambda_0} x_t = \Psi_{\lambda_0} y(t)$, where $x_t = x_t(\psi_c)$ is a solution of (2.2) with $x_0 = \psi_c \in \mathscr{C}$ then there exist $d \times n$ and $d \times d$ constant matrices $\hat{C}$ and $\hat{B}$ with the spectrum of $\hat{B} = \{\lambda_0\}$ and an $n \times 1$ vector function h: $\mathscr{C} \to R^n$ with $h(\psi_c) = (f(\psi_n(-r)) - f'(0)\,\psi_n(-r), 0, \ldots, 0)^T$, f and $f'(0)$ as before, such that $\dot{y}(t) = \hat{B}y(t) + \hat{C}h(x_t)$. However, it is easily seen that $h(\psi_c)$ depends only on the nth component of ψ_c. Define $\hat{h}(\psi) = (f(\psi_n(-r)) - f'(0)\,\psi_n(-r), 0, \ldots, 0)^T$, then for ψ_c such that $\mathscr{P}\psi_c = \psi \in \mathscr{C}_0$ we have $h(\psi_c) = \hat{h}(\psi)$.

Let $\psi_c \in \mathscr{C}$ be any ψ_c such that $\mathscr{P}(\psi_c) = \psi \in \mathscr{C}_0$ and follow Hale [10, p. 232]. Let $\mathscr{P}(\pi_{\lambda_0} x_t) = \mathscr{P}(\Psi_{\lambda_0})\, y(t)$ with $y(t)$ satisfying $\dot{y}(t) = \hat{B}y(t) + \hat{C}\hat{h}(z_t)$, where $\hat{B}$, $\hat{C}$, $\hat{h}$, z_t as before. Suppose D is a $d \times d$ positive definite symmetric matrix satisfying $\hat{B}^T D + D\hat{B} = I$ and define $V(\psi) = \hat{b}^T D \hat{b}$, where $\mathscr{P}(\pi_{\lambda_0}\psi_c) = \mathscr{P}(\Psi_{\lambda_0})\hat{b}$. If $g(\psi) = \hat{C}\hat{h}(\psi)$, then

$$\dot{V}^*(\psi) = \hat{b}^T \hat{b} + 2g^T D \hat{b}.$$

Let $\beta^2 = \min\{\hat{b}^T D \hat{b}: |\hat{b}| = 1\}$ and $\gamma = \max\{\hat{b}^T D \hat{b}: |\hat{b}| = 1\}$. Suppose η: $[0, \infty) \to R$ is a continuous nondecreasing function, $\eta(0) = 0$, such that $|g(\psi)| \leqslant \eta(\delta)\,|\psi|$ for $|\psi| \leqslant \delta$. By examining the higher order terms of the Maclaurin series in (4.3), similar to the analysis of the linear coefficient we find that the quadratic coefficient dominates the terms of order $\geqslant 3$ in some sufficiently small neighborhood of the origin. Thus, there exists $\bar{\eta} > |\hat{C}|\,|f''(0)|$ and $\delta > 0$ such that $|g(\psi)| \leqslant |\hat{C}|\,|\hat{h}(\psi)| \leqslant \bar{\eta}\,|\psi|^2 \leqslant \bar{\eta}\delta\,|\psi| = \eta(\delta)\,|\psi|$ for $|\psi| \leqslant \delta$. Given $p > 0$, choose δ_0 such that $4\gamma\,|D|\,\eta(\delta_0) < p\beta$; then as long as $|\psi| \leqslant \delta$, $0 < \delta \leqslant \delta_0$, and $V(\psi) \geqslant p^2\delta^2$, we have

$$\dot{V}^*(\psi) \geqslant \frac{1}{2\gamma} V(\psi) > 0.$$

We thus have the desired functional $V: \mathscr{C}_0 \to R^1$ and can complete the proof of Proposition 5.3 using arguments similar to those of Hale in his proof (see [10], p. 251).

Condition (ii) of Proposition 5.2 implies that

$$\nu^2 \equiv \inf\{V(\psi/\|\psi\|_{\mathscr{C}_0}): \psi \in K_0, \|\psi\|_{\mathscr{C}_0} \neq 0\} > 0$$

since there is a $\beta > 0$ such that $|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}|^2 = |\hat{b}|^2 \leqslant \beta^{-2} V(\psi)$. Now we have the proper machinery for showing ejectivity in the space $\mathscr{C}_0$, and the rest of the proof follows easily with minor notational changes as in Hale [10, p. 251]. Note that the proof of Hale only uses that τ is bounded continuous. Q.E.D.

Thus, we have seen that in order to show ejectivity in our situation we can attempt to verify that there exists $\delta > 0$ such that

$$\inf\{|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}|: \psi \in K_0, \|\psi\|_{\mathscr{C}_0} = \delta\} > 0$$

Let us elaborate on this condition and the quantities it involves for our example. Assume $\lambda \in R^+$, then

$$(b + \lambda)\left[\prod_{j=2}^{n} (\beta_j + \lambda)\right] - \left[\prod_{j=2}^{n} \alpha_j\right] f'(0)\, e^{-\lambda r} > 0$$

and thus by (2.6) we see that if λ_0 is an eigenvalue such that $\operatorname{Re}\lambda_0 > 0$ then we must have $\operatorname{Im}(\lambda_0) \neq 0$. Let $\lambda_0 = \mu_0 + i\sigma_0$ be an eigenvalue such that λ_0 lies inside Γ where Γ is as in Section 3; then we can assume $\mu_0 > 0$ and $0 < \sigma_0 < \pi/r$.

We shall now find a basis Φ_{λ_0} of (2.5) assuming λ_0 is simple. Let

$$\eta(\theta) = e^{-\lambda_0\theta}(b_1, \ldots, b_n) \qquad \text{for} \quad \theta \in [-r, 0],$$

then

$$\dot{\eta}(\theta) = -\lambda_0 e^{-\lambda_0\theta}(b_1, \ldots, b_n),$$

and

$$\begin{aligned}
-\eta(\theta)A - \eta(\theta + r)B &= -\,e^{-\lambda_0\theta}(b_1, \ldots, b_n)A - e^{-\lambda_0\theta}e^{-\lambda_0 r}(b_1, \ldots, b_n)B \\
&= -\,e^{-\lambda_0\theta}(b_1, \ldots, b_n)\left\{\begin{pmatrix} -b & 0 & \cdots & \cdots & 0 \\ \alpha_2 & -\beta_2 & \cdot & & \vdots \\ 0 & \cdot & \cdot & \cdot & \vdots \\ \vdots & & \cdot & \cdot & 0 \\ 0 & \cdots & 0 & \alpha_n & -\beta_n \end{pmatrix}\right. \\
&\qquad \left. -\, e^{-\lambda_0 r}\begin{pmatrix} 0 & \cdots & 0 & f'(0) \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\right\}.
\end{aligned}$$

Substituting these into (2.5) and multiplying by $e^{\lambda_0\theta}$ we see that

$$-\lambda_0(b_1,\ldots,b_n) = -(-bb_1+\alpha_2b_2, -\beta_2b_2+\alpha_3b_3,\ldots, -\beta_ib_i+\alpha_{i+1}b_{i+1},\ldots, \\ -\beta_nb_n+b_1f'(0)\,e^{-\lambda_0r}).$$

So componentwise we obtain the following equations:

$$\lambda_0b_1+bb_1-\alpha_2b_2=0,$$
$$\lambda_0b_i+\beta_ib_i-\alpha_{i+1}b_{i+1}=0, \qquad i=2,\ldots,n-1,$$
$$\lambda_0b_n+\beta_nb_n-e^{-\lambda_0r}f'(0)\,b_1=0,$$

which imply that

$$\begin{aligned} b_1 &= \frac{\alpha_2}{b+\lambda_0}\,b_2, \\ b_i &= \frac{\alpha_{i+1}}{\beta_i+\lambda_0}\,b_{i+1}, \qquad i=2,\ldots,n-1, \\ b_n &= \frac{e^{-\lambda_0r}f'(0)}{\beta_n+\lambda_0}\,b_1. \end{aligned} \tag{5.3}$$

But (5.3) implies

$$b_1 = \frac{\alpha_2\cdots\alpha_nf'(0)\,e^{-\lambda_0r}}{(b+\lambda_0)(\beta_2+\lambda_0)\cdots(\beta_n+\lambda_0)}\,b_1,$$

so by (2.6) we have that b_1 can be chosen arbitrarily. Let $b_1=1$. Then from (5.3) we must have

$$\begin{aligned} b_2 &= (b+\lambda_0)/\alpha_2, \\ b_i &= (b+\lambda_0)\left[\prod_{j=2}^{i-1}(\beta_j+\lambda_0)\right]\Big/\left(\prod_{j=2}^{i}\alpha_j\right), \qquad i=3,\ldots,n. \end{aligned} \tag{5.4}$$

Using this we see that $\eta(s)=(\eta_1(s),\ldots,\eta_n(s))$, $\eta_j(s)=b_je^{-\lambda_0s}$ with b_j from (5.4), forms an appropriate basis for (2.5) with simple eigenvalue λ_0, so take $\Phi_{\lambda_0}=\eta(s)$ for $s\in[-r,0]$.

We shall establish that, indeed, λ_0 must be simple. Suppose that λ_0 is not simple. For ease of argument let us assume λ_0 has a multiplicity of two. In this case, we seek a solution of (2.5) of the form $\eta(s)=(\mathbf{b}s+\mathbf{c})e^{-\lambda_0s}$ with $\mathbf{b}=(b_1,\ldots,b_n)$ and $\mathbf{c}=(c_1,\ldots,c_n)$. Using this we find that

$$\dot\eta(s) = (-\lambda_0\mathbf{b}s+\mathbf{b}-\lambda_0\mathbf{c})\,e^{-\lambda_0s}$$

and

$$-\eta(s)A-\eta(s+r)B = -(\mathbf{b}s+\mathbf{c})\,e^{-\lambda_0s}A-(\mathbf{b}s+\mathbf{b}r+\mathbf{c})\,e^{-\lambda_0r}e^{-\lambda_0s}B. \tag{5.5}$$

By equating the terms in (5.5) of form $se^{-\lambda_0 s}$ we see that $\mathbf{b}$ in this case is the same as $(b_1, ..., b_n)$ computed above for λ_0 simple.

Now considering the terms in (5.5) of the form $e^{-\lambda_0 s}$ we have $\mathbf{b} - \lambda_0 \mathbf{c} = -\mathbf{c}A - (\mathbf{b}r + \mathbf{c})e^{-\lambda_0 r}B$ or equivalently

$$[(b_1, ..., b_n) - \lambda_0(c_1, ..., c_n)] = (c_1, ..., c_n) \begin{pmatrix} b & 0 & \cdots & \cdots & 0 \\ -\alpha_2 & \beta_2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha_n & \beta_n \end{pmatrix}$$

$$- re^{-\lambda_0 r}(0, ..., 0, b_1 f'(0)) - (0, ..., 0, c_1 f'(0)\, e^{-\lambda_0 r}).$$

Expanding this term by term we see that

$$\lambda_0 c_1 - b_1 + bc_1 - \alpha_2 c_2 = 0,$$

$$\lambda_0 c_i - b_i + \beta_i c_i - \alpha_{i+1} c_{i+1} = 0, \qquad i = 2, ..., n-1,$$

$$\lambda_0 c_n - b_n + \beta_n c_n - rb_1 f'(0)\, e^{-\lambda_0 r} - c_1 f'(0)\, e^{-\lambda_0 r} = 0,$$

so we find that

$$c_1 = (c_2\alpha_2 + b_1)/(b + \lambda_0),$$

$$c_i = (c_{i+1}\alpha_{i+1} + b_i)/(\beta_i + \lambda_0),$$

$$c_n = [b_n + (rb_1 + c_1) f'(0)\, e^{-\lambda_0 r}]/(\beta_n + \lambda_0).$$

Using the characteristic equation (2.6) we can also write

$$c_n = b_n/(\beta_n + \lambda_0) + \left[(rb_1 + c_1)(b + \lambda_0) \prod_{j=2}^{n-1} (\beta_j + \lambda_0)\right] \Big/ \left(\prod_{j=2}^{n} \alpha_j\right).$$

Since b_1 is arbitrary let us assume $b_1 = 1$ and so $b_i = (b + \lambda_0)[\prod_{j=2}^{i-1} (\beta_j + \lambda_0)]/(\prod_{j=2}^{i} \alpha_j)$. From this we can calculate the c_i's obtaining

$$c_n = \left(\frac{1}{\beta_n + \lambda_0} + r + c_1\right) b_n,$$

$$c_i = \left(\sum_{j=i}^{n} (1/(\beta_j + \lambda_0)) + r + c_1\right) b_i,$$

$$c_1 = \left(\frac{1}{b + \lambda_0} + \sum_{j=2}^{n} (1/(\beta_j + \lambda_0)) + r + c_1\right).$$

Using the expression for c_1 we see that

$$\left(\frac{1}{b+\lambda_0} + \sum_{j=2}^{n} (1/(\beta_j + \lambda_0)) + r\right) = 0. \tag{5.6}$$

However, since λ_0 has a positive imaginary part we obtain that the left side of Eq. (5.6) must have a negative imaginary part which is a contradiction. This shows that λ_0 cannot have a multiplicity of two. By assuming a multiplicity of $k+1$ we would seek a solution of the form $\eta(s) = (\mathbf{b}s^k + \mathbf{c}s^{k-1} + \cdots)e^{-\lambda_0 s}$. A similar calculation on the coefficients of the two highest powers of s will give similar contradiction as in the case when λ_0 was assumed to have a multiplicity of two. This shows that λ_0 is indeed simple.

Let $\psi = (\xi_1, \xi_2, ..., \xi_n)^T \in \mathscr{C}$, then (5.2a) gives a bilinear form where we may take $b_1 = 1$. Suppose $b_j = c_j + id_j$, $j = 2,..., n$, where c_j, d_j are real. Splitting the bilinear form into its real and imaginary parts we get

$$\operatorname{Re}(\eta, \psi)_{\mathscr{C}_0} = \sum_{j=1}^{n} c_j\xi_j(0) + f'(0) \int_{-r}^{0} e^{-\mu_0(\theta+r)} \cos[\sigma_0(\theta + r)]\, \xi_n(\theta)\, d\theta \tag{5.7a}$$

and

$$\operatorname{Im}(\eta, \psi)_{\mathscr{C}_0} = \sum_{j=1}^{n} d_j\xi_j(0) - f'(0) \int_{-r}^{0} e^{-\mu_0(\theta+r)} \sin[\sigma_0(\theta + r)]\, \xi_n(\theta)\, d\theta. \tag{5.7b}$$

Using these along with our previous arguments and findings we can finally establish the following theorem:

THEOREM 5.1. *Assume the following hold:*

(i) *r is sufficiently large such that* $\arctan \pi/(rb) + \sum_{j=2}^{n-1} \arctan \pi/(r\beta_j) < \pi$;

(ii) (H1);

(iii) $D(iv_0) < C_1$.

Then (2.2) *has a non-constant periodic solution.*

Proof. Condition (iii), in light of Proposition 3.1 yields the existence of an eigenvalue λ_0 with $\operatorname{Re} \lambda_0 > 0$. The hypothesis (H1) is used to establish Propositions 5.1 and 5.2 which give us the bounded continuous map τ: $K_0\backslash\{0\} \to [r, \infty)$ and $\mathscr{T}$: $K_0\backslash\{0\} \to K_0\backslash\{0\}$. We need only argue the existence of a $\delta > 0$ such that $\inf\{|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}| : \psi \in K_0, \ \|\psi\|_{\mathscr{C}_0} = \delta\} > 0$. Then Proposition 5.3 guarantees that $\{0\}$ is an ejective point in $\mathscr{C}_0$ of the map $\mathscr{T}$. Combining these results, we find the hypotheses of Theorem 2.1 satisfied and thus the conclusion yields the existence of a fixed point of the map $\mathscr{T}$. A fixed point of the map $\mathscr{T}$ implies the existence of a non-constant periodic solution.

Thus, we must examine $\inf\{|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}| : \psi \in K_0, \|\psi\|_{\mathscr{C}_0} = \delta\}$, which means (Φ_{λ_0}, ψ) must be considered. The proof follows the technique of Hale [10, p. 266]. Having computed a basis Φ_{λ_0} above it suffices to study the bilinear form given in Eqs. (5.7a) and (5.7b). Using (5.4) we see $b_n = (b + \lambda_0)[\prod_{j=2}^{n-1} (\beta_j + \lambda_0)]/(\prod_{j=2}^{n} \alpha_j)$. Since $\lambda_0 = \mu_0 + i\sigma_0$, where $0 < \sigma_0 < \pi/r$, we find that

$$\begin{aligned}\arg b_n &= \arg(b + \lambda_0)\left[\prod_{j=2}^{n-1} (\beta_j + \lambda_0)\right] \\ &= \arctan(\sigma_0/(b + \mu_0)) + \sum_{j=2}^{n-1} \arctan(\sigma_0/(\beta_j + \mu_0)) \\ &< \arctan(\pi/rb) + \sum_{j=2}^{n-1} \arctan(\pi/r\beta_j).\end{aligned}$$

Using condition (i) we see that $\arg b_n < \pi$. By looking at the arguments of each b_j given by (5.4) we can obtain

$$\arg b_i = \arctan(\sigma_0/(b + \mu_0)) + \sum_{j=2}^{i-1} \arctan(\sigma_0/(\beta_j + \mu_0)),$$

and so it is easy to see $\arg b_j < \arg b_{j+1}$ for each $j = 1,..., n - 1$, ($\arg b_1 = 0$). For $b_2,..., b_n$ we have $0 < \arg b_j < \pi$, which gives $\operatorname{Im} b_j > 0$, $j = 2,..., n$, which in turn implies $d_j > 0$ for $j = 2,..., n$, where $b_j = c_j + id_j$.

Suppose for some $\delta > 0$ there exists a sequence $\psi_k = (\mathbf{x}_0^k, \phi_n^k) \in K_0$ with $\|\psi_k\|_{\mathscr{C}_0} = \delta$ such that $|(\Phi_{\lambda_0}, \psi_k)_{\mathscr{C}_0}| \to 0$ as $k \to \infty$. This would contradict $\inf\{|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}| : \psi \in K_0, \|\psi\|_{\mathscr{C}_0} = \delta\} > 0$. We know $f'(0) < 0$ and $0 < \sigma_0 < \pi/r$ which implies $\sin[\sigma_0(\theta + r)] \geqslant 0$ for $\theta \in [-r, 0]$, and so

$$-f'(0) \int_{-r}^{0} e^{-\mu_0(\theta+r)} \sin[\sigma_0(\theta + r)]\, \xi_n(\theta)\, d\theta \geqslant 0.$$

Since each $d_j > 0$ for $j \geqslant 2$, if $|(\Phi_{\lambda_0}, \psi_k)_{\mathscr{C}_0}| \to 0$ then its real and imaginary parts $\to 0$ and so (5.7b) implies $\xi_j^k(0) \to 0$ for $j \geqslant 2$. But $e^{\beta_n t}\xi_n(t)$ is nonnegative nondecreasing in t on $[-r, 0]$ and $\xi_n^k(0) \to 0$, so we find $\xi_n^k(t) \to 0$ uniformly on $[-r, 0]$. Since $b_1 = c_1 = 1$, from (5.7a) and the information on ψ_k from (5.7b) we see $\xi_1^k(0) \to 0$. Using this knowledge we see that $\mathbf{x}_0^k \to 0$ and $\phi_n^k \to 0$, which contradicts $\|\psi_k\|_{\mathscr{C}_0} = \delta$. Therefore there exists $\delta > 0$ such that $\inf\{|(\Phi_{\lambda_0}, \psi)_{\mathscr{C}_0}| : \psi \in K_0, \|\psi\|_{\mathscr{C}_0} = \delta\} > 0$. Q.E.D.

COROLLARY 5.1. *Assume the following hold*: (i) $n = 3$; (ii) (H1); (iii) $D(i\nu_0) < C_1$. *Then* (2.2) *has a non-constant periodic solution.*

Proof. For $n = 3$ we find the hypothesis (i) of Theorem 5.1 holds for all r. Q.E.D.

COROLLARY 5.2. *Suppose* $b(\prod_{j=2}^{n} \beta_j) < -f'(0)(\prod_{j=2}^{n} \alpha_j)$. *Then there exists an* r *sufficiently large such that* (2.2) *has a non-constant periodic solution.*

Proof. It is easy to see that there exists an r_1 such that hypothesis (i) of Theorem 5.1 holds for all $r > r_1$. From our comments after Proposition 3.1 we see that since $b(\prod_{j=2}^{n} \beta_j) < -f'(0)(\prod_{j=2}^{n} \alpha_j)$ there exists an r_2 such that $r > r_2$ implies $D(iv_0) < C_1$. And finally by our comments after the statement of (H1) we observed that there exists an r_3 such that $r > r_3$ implies f satisfies (H1). Let $\bar{r} = \max\{r_1, r_2, r_3\}$, then $r > \bar{r}$ implies the hypotheses of Theorem 5.1 hold. Q.E.D.

6. CONCLUDING REMARKS

Goodwin [8] suggested that negative feedback or repression in biosynthetic pathways might be used to explain experimentally observed epigenetic oscillations in living organisms. The evidence for epigenetic oscillations in prokaryotes is not unequivocal. Even if oscillations do arise, they may or may not be the result of a repressible system. (An asynchronously grown cell culture showed only damped oscillations, and this was for a positive feedback or inducible system [13, 14].) In the repression models Goodwin conjectured that inserting delays could have a destablilizing effect and result in oscillations. Banks and Mahaffy [2] showed that for a large class of repression models (the case when $\rho = 1$ in (2.2)), the introduction of delays, whether discrete or distributed, will not produce oscillations. In fact, this particular class of models has a unique asymptotically stable equilibrium.

Our results in this paper show conclusively that there are ranges of parameters for which the system (2.2) has periodic solutions. Computer simulations for certain parameter values did seem to approach a stable limit cycle. Using our knowledge of the parameters from the derivation of the model, we inferred a range of biologically significant parameter values (see Banks and Mahaffy [3]). Using these values, we found that in our computer simulations the solutions approached the equilibrium solution rather than oscillating. (The delays required to produce oscillations were too large to be of biological significance.) However, computer simulations can be misleading as was demonstrated by Banks and Mahaffy [4]. In [4] we proved that a cyclic gene repression model was asymptotically stable even though other investigators had conjectured, based on computer results, that sustained oscillations could arise. This emphasizes the importance of an analytical proof of oscillations whenever it is possible.

The results of this paper give valuable insight into the destabilizing effects of the parameters on the system (2.2). In particular, the reader is referred to the linear analysis of Section 3. This knowledge shows the relative importance of the parameters in our system and so can be useful in predicting the behavior of modifications of (2.2). For the system (2.2) we believe, based on the analysis presented here and in [3], that other biological factors must be taken into account before sustained oscillations of biological significance could arise in the models.

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