7.10.1.e. (15 pts) Solve $\frac{d^2u}{dt^2} = c^2\nabla^2 u$ inside sphere of radius a, so

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right).$$

The BC is $u(a,\theta,\phi,t) = 0$ with ICs $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos(3\theta)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$. The implicit BCs are $|u(0,\theta,\phi,t)| < \infty$, $u(\rho,-\pi,\phi,t) = u(\rho,\pi,\phi,t)$, $\frac{\partial u}{\partial\theta}(\rho,-\pi,\phi,t) = \frac{\partial u}{\partial\theta}(\rho,\pi,\phi,t)$, $|u(\rho,\theta,0,t)| < \infty$, and $|u(\rho,\theta,\pi,t)| < \infty$.

We apply separation of variables by letting $u(\rho,\theta,\phi,t) = h(t)f(\rho)q(\theta)g(\phi)$. Initially, we have:

$$\frac{h''}{c^2h} = \frac{\nabla^2(fgq)}{fgq} = -\lambda, \qquad \text{so} \qquad h'' + c^2\lambda h = 0.$$

For $\lambda > 0$, the solution of the t-equation is:

$$h(t) = A\cos\left(c\sqrt{\lambda}t\right) + B\sin\left(c\sqrt{\lambda}t\right).$$

Since the initial velocity is zero, $\frac{dh}{dt}(0) = 0$, which implies $B = 0$, so $h(t) = A\cos\left(c\sqrt{\lambda}t\right)$.

The spatial variables give:

$$\frac{gq}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \frac{fq}{\rho^2\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \frac{fg}{\rho^2\sin^2\phi}\frac{d^2q}{d\theta^2} + \lambda fgq = 0,$$

so we multiply by $\rho^2\sin^2\phi$ and divide by fgq. The result satisfies:

$$\frac{q''}{q} = -\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) - \frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) - \lambda\rho^2\sin^2\phi = -\mu.$$

The first Sturm-Liouville problem is:

$$q'' + \mu q = 0, \quad \text{with} \quad q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0 = 0$ with eigenfunction $q_0(\theta) = 1$ and eigenvalues $\mu_m = m^2$ with eigenfunctions $q_m(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta)$.

By dividing by $\sin^2\phi$ with $\mu = m^2$, we can write:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \rho^2\lambda = -\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \frac{m^2}{\sin^2\phi} = \nu,$$

which gives the remaining two Sturm-Liouville problems.

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \left(\lambda\rho^2 - \nu\right)f = 0$$

and

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\nu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0.$$

First we solve the SL Problem in ϕ by letting $x = \cos\phi$ (with $0 < \phi < \pi$ or $-1 < x < 1$). From the chain rule we have:

$$\frac{dg}{d\phi} = \frac{dg}{dx}\frac{dx}{d\phi} = -\sin\phi\frac{dg}{dx} = -\sqrt{1-x^2}\frac{dg}{dx}.$$

It follows that the SL Problem in ϕ is transformed to an ODE in x by

$$-\sin\phi\frac{d}{dx}\left(\sin\phi\left(-\sin\phi\frac{dg}{d\phi}\right)\right) + \sin\phi\left(\nu - \frac{m^2}{\sin^2\phi}\right)g = 0$$

or Legendre's equation:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{d\phi}\right) + \left(\nu - \frac{m^2}{1-x^2}\right)g = 0,$$

which has the BCs $g(1)$ and $g(-1)$ are bounded. This has eigenvalues $\nu_n = n(n+1)$, giving the general solution:

$$g(x) = c_1 P_n^m(x) + c_2 Q_n^m(x),$$

which are associated Legendre functions. Only the polynomial $P_n^m(x)$ is bounded at $x = \pm 1$, which gives the eigenfunctions:

$$g_{mn}(x) = P_n^m(x) \qquad \text{or} \qquad g_{mn}(\phi) = P_n^m(\cos\phi), \qquad m = 0, 1, 2, ... \quad \text{and} \quad n \geq m.$$

The orthogonality condition for these associated Legendre polynomials is:

$$\int_0^\pi P_n^m(\cos\phi)P_p^m(\cos\phi)\sin\phi\, d\phi = 0, \qquad n \neq p.$$

The radial SL-problem (ρ) has $\nu = n(n+1)$, so we have:

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \left(\lambda\rho^2 - n(n+1)\right)f = 0, \qquad \text{with} \quad n \geq m.$$

This is again a form of Bessel's equation producing spherical Bessel functions. The only bounded solution at $\rho = 0$ is:

$$f(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda}\rho\right).$$

If the k^{th} zero of the $n + \frac{1}{2}$ spherical Bessel function is denoted z_{nk} (so $J_{n+\frac{1}{2}}(z_{nk}) = 0$), then the eigenvalues and eigenfunctions are:

$$\lambda_{nk} = \left(\frac{z_{nk}}{a}\right)^2 \qquad \text{and} \qquad f_{nk}(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right), \quad n \geq m \quad \text{and} \quad k = 1, 2, ...$$

The orthogonality condition for these spherical Bessel functions is:

$$\int_0^a J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)\rho\, d\rho = 0, \qquad j \neq k.$$

The Superposition principle gives:

$$\begin{aligned} u(\rho,\theta,\phi,t) &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} A_{k0n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)P_n^0(\cos\phi)\cos\left(c\sqrt{\lambda_{nk}}t\right) + \\ & \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(A_{kmn}\cos(m\theta)+B_{kmn}\sin(m\theta)\right)\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)\times \\ & \qquad P_n^m(\cos\phi)\cos\left(c\sqrt{\lambda_{nk}}t\right). \end{aligned}$$

The initial position is given by $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos 3\theta$, so our orthogonality conditions can be readily applied to show that

$$B_{kmn} = 0 \quad \text{for all} \quad k,m,n, \qquad \text{and} \qquad A_{kmn} = 0 \quad \text{for all} \quad k, m \neq 3, n.$$

We have

$$F(\rho,\phi) = \sum_{k=1}^{\infty}\sum_{n=3}^{\infty} A_{k3n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)P_n^3(\cos\phi),$$

where

$$A_{k3n} = \frac{\int_0^a\int_0^\pi F(\rho,\phi)J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)P_n^3(\cos\phi)\sin\phi\,\rho^{3/2}d\phi\,d\rho}{\int_0^\pi\left(P_n^3(\cos\phi)\right)^2\sin\phi\,d\phi\int_0^a\rho\left(J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)\right)^2 d\rho}.$$

Thus, the solution satisfies:

$$u(\rho,\theta,\phi,t) = \sum_{k=1}^{\infty}\sum_{n=3}^{\infty} A_{k3n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nk}}\rho\right)\cos(3\theta)P_n^3(\cos\phi)\cos\left(c\sqrt{\lambda_{nk}}t\right).$$

7.10.2.c. (15 pts) Solve $\frac{du}{dt} = k\nabla^2 u$ inside sphere of radius a, so

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right).$$

The BC is $u(a,\theta,\phi,t) = 0$ with IC $u(\rho,\theta,\phi,0) = F(\rho,\theta)\cos\theta$. The implicit BCs are $|u(0,\theta,\phi,t)| < \infty$, $u(\rho,-\pi,\phi,t) = u(\rho,\pi,\phi,t)$, $\frac{\partial u}{\partial\theta}(\rho,-\pi,\phi,t) = \frac{\partial u}{\partial\theta}(\rho,\pi,\phi,t)$, $|u(\rho,\theta,0,t)| < \infty$, and $|u(\rho,\theta,\pi,t)| < \infty$.

This analysis parallels the previous problem. We apply separation of variables by letting $u(\rho,\theta,\phi,t) = h(t)f(\rho)q(\theta)g(\phi)$. Initially, we have:

$$\frac{h'}{kh} = \frac{\nabla^2(fgq)}{fgq} = -\lambda, \qquad \text{so} \qquad h' + k\lambda h = 0.$$

For $\lambda > 0$, the solution of the t-equation is:

$$h(t) = A\,e^{-k\lambda t}.$$

The spatial variables give:

$$\frac{gq}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\frac{fq}{\rho^2\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\frac{fg}{\rho^2\sin^2\phi}\frac{d^2q}{d\theta^2}+\lambda fgq=0,$$

so we multiply by $\rho^2\sin^2\phi$ and divide by fgq. The result satisfies:

$$\frac{q''}{q}=-\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)-\frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)-\lambda\rho^2\sin^2\phi=-\mu.$$

The first Sturm-Liouville problem is:

$$q''+\mu q=0,\quad \text{with}\quad q(-\pi)=q(\pi)\quad\text{and}\quad q'(-\pi)=q'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0=0$ with eigenfunction $q_0(\theta)=1$ and eigenvalues $\mu_m=m^2$ with eigenfunctions $q_m(\theta)=A_m\cos(m\theta)+B_m\sin(m\theta)$.

By dividing by $\sin^2\phi$ with $\mu=m^2$, we can write:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\rho^2\lambda=-\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\frac{m^2}{\sin^2\phi}=\nu,$$

which gives the remaining two Sturm-Liouville problems.

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right)+\left(\lambda\rho^2-\nu\right)f=0$$

and

$$\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\nu\sin\phi-\frac{m^2}{\sin\phi}\right)g=0.$$

First we solve the SL Problem in ϕ by letting $x=\cos\phi$ (with $0<\phi<\pi$ or $-1<x<1$). From the chain rule we have:

$$\frac{dg}{d\phi}=\frac{dg}{dx}\frac{dx}{d\phi}=-\sin\phi\frac{dg}{dx}=-\sqrt{1-x^2}\frac{dg}{dx}.$$

It follows that the SL Problem in ϕ is transformed to an ODE in x by

$$-\sin\phi\frac{d}{dx}\left(\sin\phi\left(-\sin\phi\frac{dg}{d\phi}\right)\right)+\sin\phi\left(\nu-\frac{m^2}{\sin^2\phi}\right)g=0$$

or Legendre's equation:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{d\phi}\right)+\left(\nu-\frac{m^2}{1-x^2}\right)g=0,$$

which has the BCs $g(1)$ and $g(-1)$ are bounded. This has eigenvalues $\nu_n=n(n+1)$, giving the general solution:

$$g(x)=c_1P_n^m(x)+c_2Q_n^m(x),$$

which are associated Legendre functions. Only the polynomial $P_n^m(x)$ is bounded at $x=\pm1$, which gives the eigenfunctions:

$$g_{mn}(x)=P_n^m(x)\qquad\text{or}\qquad g_{mn}(\phi)=P_n^m(\cos\phi),\qquad m=0,1,2,...\quad\text{and}\quad n\geq m.$$

The orthogonality condition for these associated Legendre polynomials is:

$$\int_0^\pi P_n^m(\cos\phi)P_p^m(\cos\phi)\sin\phi\,d\phi = 0, \qquad n \neq p.$$

The radial SL-problem (ρ) has $\nu = n(n+1)$, so we have:

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + \left(\lambda\rho^2 - n(n+1)\right)f = 0, \qquad \text{with} \quad n \geq m.$$

This is again a form of Bessel's equation producing spherical Bessel functions. The only bounded solution at $\rho = 0$ is:

$$f(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda}\rho\right).$$

If the j^{th} zero of the $n+\frac{1}{2}$ spherical Bessel function is denoted z_{nj} (so $J_{n+\frac{1}{2}}(z_{nj}) = 0$), then the eigenvalues and eigenfunctions are:

$$\lambda_{nj} = \left(\frac{z_{nj}}{a}\right)^2 \qquad \text{and} \qquad f_{nj}(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right), \quad n \geq m \quad \text{and} \quad j = 1, 2, ...$$

The orthogonality condition for these spherical Bessel functions is:

$$\int_0^a J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{ni}}\rho\right)J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)\rho\,d\rho = 0, \qquad i \neq j.$$

The Superposition principle gives:

$$\begin{aligned} u(\rho,\theta,\phi,t) &= \sum_{j=1}^\infty\sum_{n=0}^\infty A_{j0n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)P_n^0(\cos\phi)e^{-k\lambda_{nj}t} + \\ &\quad \sum_{j=1}^\infty\sum_{m=1}^\infty\sum_{n=m}^\infty \left(A_{jmn}\cos(m\theta) + B_{jmn}\sin(m\theta)\right)\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right) \times \\ &\qquad\qquad P_n^m(\cos\phi)e^{-k\lambda_{nj}t}. \end{aligned}$$

The initial position is given by $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos\theta$, so our orthogonality conditions can be readily applied to show that

$$B_{jmn} = 0 \quad \text{for all} \quad j, m, n, \qquad \text{and} \qquad A_{jmn} = 0 \quad \text{for all} \quad j, m \neq 1, n.$$

We have

$$F(\rho,\phi) = \sum_{j=1}^\infty\sum_{n=1}^\infty A_{j1n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)P_n^1(\cos\phi),$$

where

$$A_{j1n} = \frac{\int_0^a\int_0^\pi F(\rho,\phi)J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)P_n^1(\cos\phi)\sin\phi\,\rho^{3/2}d\phi\,d\rho}{\int_0^\pi\left(P_n^1(\cos\phi)\right)^2\sin\phi\,d\phi\int_0^a\rho\left(J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)\right)^2d\rho}.$$

Thus, the solution satisfies:

$$u(\rho,\theta,\phi,t) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} A_{j1n}\rho^{-1/2} J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{nj}}\rho\right)\cos(\theta) P_n^1(\cos\phi) e^{-k\lambda_{nj}t}.$$

7.10.8 (10 pts) The ODE related to Bessel's equation is:

$$x^2\frac{d^2 f}{dx^2} + x(1-2a-2bx)\frac{df}{dx} + \left(a^2 - p^2 + (2a-1)bx + (d^2+b^2)x^2\right) f = 0. \qquad (1)$$

When $f(x) = x^a e^{bx} Z_p(dx)$, we want to find parameters a, b, d, and p, so $Z_p(x)$ solves Bessel's equation:

$$x^2 {Z_p}'' + x{Z_p}' + (x^2 - p^2) Z_p = 0.$$

With this form of $f(x)$, we have:

$$\begin{aligned} f'(x) &= ax^{a-1}e^{bx}Z_p(dx) + x^a b e^{bx} Z_p(dx) + x^a e^{bx} d{Z_p}'(dx) \\ &= x^{a-1}e^{bx}\left((a+bx)Z_p(dx) + dx{Z_p}'(dx)\right), \end{aligned}$$

and

$$\begin{aligned} f''(x) &= a(a-1)x^{a-2}e^{bx}Z_p(dx) + 2abx^{a-1}e^{bx}Z_p(dx) + b^2x^a e^{bx} dZ_p(dx) \\ &\quad +2adx^{a-1}e^{bx}{Z_p}'(dx) + 2bdx^a e^{bx}{Z_p}'(dx) + d^2x^a e^{bx}{Z_p}''(dx) \\ &= x^{a-2}e^{bx}\left((a(a-1) + 2abx + b^2x^2)Z_p(dx) + \right. \\ &\quad \left. 2d(ax+bx^2){Z_p}'(dx) + d^2x^2{Z_p}''(dx)\right). \end{aligned}$$

These are substituted into Eq. (1) giving:

$$\begin{aligned} x^a e^{bx}\left((a(a-1) + 2abx + b^2x^2)Z_p(dx) + 2d(ax+bx^2){Z_p}'(dx) + d^2x^2{Z_p}''(dx)\right) \\ (1-2a-2bx)x^a e^{bx}\left((a+bx)Z_p(dx) + dx{Z_p}'(dx)\right) \\ \left(a^2 - p^2 + (2a-1)bx + (d^2+b^2)x^2\right) x^a e^{bx} Z_p(dx) &= 0, \end{aligned}$$

which after cancellation is equivalent to

$$d^2x^2{Z_p}''(dx) + dx{Z_p}'(dx) + (d^2x^2 - p^2)Z_p(dx) = 0.$$

For $y = dx$, this has the solution $Z_p(y)$ to Bessel's equation.

The radial SL-problem for the spherical problem has the form:

$$x^2 f'' + 2xf' + (\lambda x - n(n+1)) f = 0.$$

For Eq. (1) to have the same form as the spherical ODE, we need $1 - 2a - 2bx = 2$, so $b = 0$ and $a = -\frac{1}{2}$ from the coefficient of f'. From the coefficient of f with a and b above, we need $d^2 = \lambda$ or $d = \sqrt{\lambda}$ and $\frac{1}{4} - p^2 = -n(n+1)$ or $p^2 = n^2 + n + \frac{1}{4} = \left(n + \frac{1}{2}\right)^2$ or $p = n + \frac{1}{2}$. It follows that

$$f(x) = x^{-1/2}e^{0x}J_{n+\frac{1}{2}}\left(\sqrt{\lambda}x\right) = x^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda}x\right),$$

which matches the solution for the spherical Bessel function.

7.10.9.a (10 pts) Consider Laplace's equation inside a sphere $\nabla^2 u = 0$ of radius a, so

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

The BC's are $u(a, \theta, \phi, t) = F(\phi)\cos(4\theta)$ with implicit BCs $|u(0, \theta, \phi, t)| < \infty$, $u(\rho, -\pi, \phi, t) = u(\rho, \pi, \phi, t)$, $\frac{\partial u}{\partial \theta}(\rho, -\pi, \phi, t) = \frac{\partial u}{\partial \theta}(\rho, \pi, \phi, t)$, $|u(\rho, \theta, 0, t)| < \infty$, and $|u(\rho, \theta, \pi, t)| < \infty$.

This analysis parallels the first two problems. We apply separation of variables by letting $u(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi)$. We have:

$$\frac{gq}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \frac{fq}{\rho^2 \sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \frac{fg}{\rho^2 \sin^2\phi}\frac{d^2 q}{d\theta^2} = 0,$$

so we multiply by $\rho^2 \sin^2\phi$ and divide by fgq. The result satisfies:

$$\frac{q''}{q} = -\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) - \frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) = -\mu.$$

The first Sturm-Liouville problem is:

$$q'' + \mu q = 0, \quad \text{with} \quad q(-\pi) = q(\pi) \quad \text{and} \quad q'(-\pi) = q'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0 = 0$ with eigenfunction $q_0(\theta) = 1$ and eigenvalues $\mu_m = m^2$ with eigenfunctions $q_m(\theta) = A_m \cos(m\theta) + B_m \sin(m\theta)$.

By dividing by $\sin^2\phi$ with $\mu = m^2$, we can write:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) = -\frac{1}{g \sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \frac{m^2}{\sin^2\phi} = \nu,$$

which gives the remaining two ODEs.

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) - \nu f = 0$$

and

$$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\nu \sin\phi - \frac{m^2}{\sin\phi}\right) g = 0.$$

First we solve the SL Problem in ϕ by letting $x = \cos\phi$ (with $0 < \phi < \pi$ or $-1 < x < 1$). From the chain rule we have:

$$\frac{dg}{d\phi} = \frac{dg}{dx}\frac{dx}{d\phi} = -\sin\phi \frac{dg}{dx} = -\sqrt{1 - x^2}\frac{dg}{dx}.$$

It follows that the SL Problem in ϕ is transformed to an ODE in x by

$$-\sin\phi \frac{d}{dx}\left(\sin\phi \left(-\sin\phi \frac{dg}{d\phi}\right)\right) + \sin\phi \left(\nu - \frac{m^2}{\sin^2\phi}\right) g = 0$$

or Legendre's equation:

$$\frac{d}{dx}\left((1 - x^2)\frac{dg}{d\phi}\right) + \left(\nu - \frac{m^2}{1 - x^2}\right) g = 0,$$

which has the BCs $g(1)$ and $g(-1)$ are bounded. This has eigenvalues $\nu_n = n(n+1)$, giving the general solution:

$$g(x) = c_1 P_n^m(x) + c_2 Q_n^m(x),$$

which are associated Legendre functions. Only the polynomial $P_n^m(x)$ is bounded at $x = \pm 1$, which gives the eigenfunctions:

$$g_{mn}(x) = P_n^m(x) \qquad \text{or} \qquad g_{mn}(\phi) = P_n^m(\cos\phi), \qquad m = 0, 1, 2, ... \quad \text{and} \quad n \geq m.$$

The orthogonality condition for these associated Legendre polynomials is:

$$\int_0^\pi P_n^m(\cos\phi) P_p^m(\cos\phi) \sin\phi \, d\phi = 0, \qquad n \neq p.$$

The radial problem (ρ) has $\nu = n(n+1)$, so we have:

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) - n(n+1))f = 0, \qquad \text{with} \quad n = 1, 2, ...$$

This is a Cauchy-Euler equation, so try a solution of the form $f(\rho) = \rho^r$. It follows that:

$$\rho^2 r(r-1)\rho^{r-2} + 2\rho r \rho^{r-1} - n(n+1)\rho^r = 0,$$

which gives the auxiliary equation $r^2 + r - n(n+1) = 0$, so $r = n$ or $-(n+1)$. Thus, the solution is:

$$f(\rho) = c_1 \rho^n + c_2 \rho^{-(n+1)}.$$

The bounded BC implies $c_2 = 0$, so $f_n(\rho) = \rho^n$.

The Superposition principle gives:

$$u(\rho, \theta, \phi) = \sum_{n=1}^{\infty} A_{0n} \rho^n P_n^0(\cos\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) \rho^n P_n^m(\cos\phi).$$

The nonhomogeneous BC gives:

$$\begin{aligned} u(a, \theta, \phi) &= \sum_{n=1}^{\infty} A_{0n} a^n P_n^0(\cos\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) a^n P_n^m(\cos\phi) \\ &= F(\phi)\cos(4\theta). \end{aligned}$$

Orthogonality gives $B_{mn} = 0$ for all m and n and $A_{mn} = 0$ for $m \neq 4$. It follows that:

$$F(\phi) = \sum_{n=4}^{\infty} A_{4n} a^n P_n^4(\cos\phi).$$

The Fourier coefficients are:

$$A_{4n} = \frac{\int_0^\pi F(\phi) P_n^4(\cos\phi)\sin\phi\, d\phi}{a^n \int_0^\pi \left(P_n^4(\cos\phi)\right)^2 \sin\phi\, d\phi}.$$

Thus, the solution satisfies:

$$u(\rho, \theta, \phi) = \rho^n \cos(4\theta) \sum_{n=4}^{\infty} A_{4n} P_n^4(\cos\phi).$$