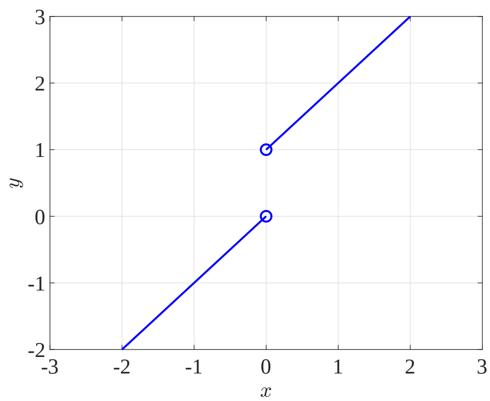
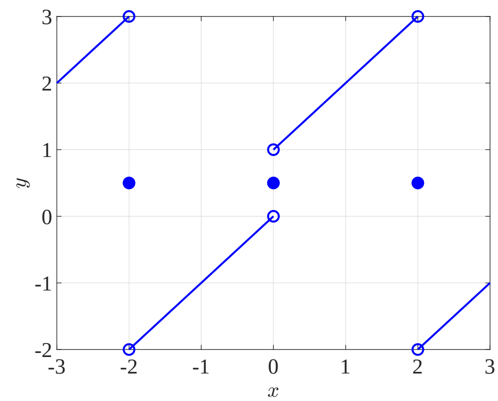
3.3.1(c) (10pts) Sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$.

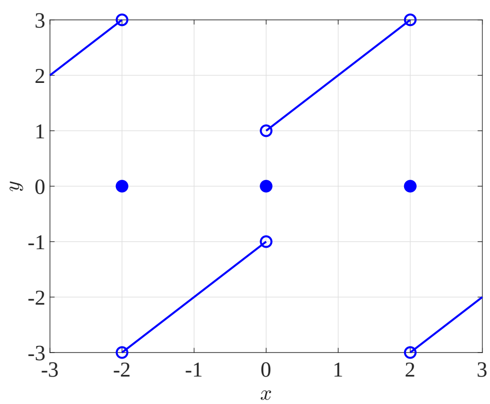
$$f(x) = \begin{cases} x, & x < 0, \\ 1 + x, & x > 0. \end{cases}$$



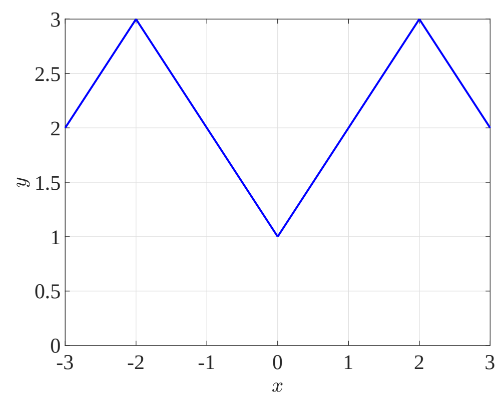
$f(x)$



Fourier series, $f(x)$



Fourier sine series, $f(x)$



Fourier cosine series, $f(x)$

3.3.14 a (15pts) Consider a function $f(x)$ that is even around $x = \frac{L}{2}$. Show that the even coefficients (n odd)of the Fourier cosine series of $f(x)$ on $0 \le x \le L$ are zero.

Proof: The odd coefficients of the Fourier cosine series are:

$$a_{2k+1} = \frac{2}{L}\int_0^L f(x)\cos\left(\tfrac{(2k+1)\pi x}{L}\right)\,dx = \frac{2}{L}\int_{-L/2}^{L/2} f(s+L/2)\cos\left(\tfrac{(2k+1)\pi(s+L/2)}{L}\right)\,ds,$$

which results from the translation, $x = s + L/2$. By trig identities, it follows that:

$$a_{2k+1} = \frac{2}{L}\int_{-L/2}^{L/2} f(s+L/2)\left[\cos\left(\tfrac{(2k+1)\pi s}{L}\right)\cos\left(\tfrac{(2k+1)\pi}{2}\right) - \sin\left(\tfrac{(2k+1)\pi s}{L}\right)\sin\left(\tfrac{(2k+1)\pi}{2}\right)\right]\,ds.$$

However, $\cos\left(\tfrac{(2k+1)\pi}{2}\right) = 0$ and $\sin\left(\tfrac{(2k+1)\pi}{2}\right) = (-1)^{k+1}$, so

$$a_{2k+1} = \frac{2}{L}\int_{-L/2}^{L/2}(-1)^k f(s+L/2)\sin\left(\tfrac{(2k+1)\pi s}{L}\right)\,ds.$$

In the integral above, the function f is even about $L/2$, which was translated to the origin, and the sine function is odd about the origin. The product of this even and odd function is an odd function, so the symmetric integral about the origin is zero. It follows that:

$$a_{2k+1} = 0.$$

which completes the proof. *q.e.d.*

b. On $0 \le x \le \frac{L}{2}$, the cosine series of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L/2} = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right).$$

The Fourier coefficients are

$$a_n = \frac{2}{L/2}\int_0^{L/2} f(x)\cos\left(\frac{2n\pi x}{L}\right)\,dx = \frac{4}{L}\int_0^{L/2} f(x)\cos\left(\frac{2n\pi x}{L}\right)\,dx.$$

Since $f(x)$ is even around $\frac{L}{2}$,

$$\begin{aligned}
\frac{4}{L}\int_0^{L/2} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx &= \frac{2}{L}\int_0^{L/2} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx + \frac{2}{L}\int_0^{L/2} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_0^{L/2} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{L/2}^{L} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_0^{L} f(x)\cos\left(\tfrac{2n\pi x}{L}\right)\,dx
\end{aligned}$$

This last term gives the even coefficients of the original series for $x \in [0, L]$. Since these series are the same, it follows that the odd coefficients of the original series must be zero.

3.4.6. (10pts) We assume that

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right).$$

There is no problem differentiating this series term-by-term to obtain a sine series:

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\left(\tfrac{n\pi x}{L}\right).$$

The error is in the second step, differentiating the Fourier sine series of e^x because the sine series is not continuous at $x = 0$ and $e^x \neq 0$ at $x = 0$ and $x = L$. The text has a generalization in Section 3.4 (Eqn. 3.4.13) for differentiation of the sine series when the derivative is only piecewise smooth. Since $\frac{d}{dx}(e^x) = e^x$, the formula gives:

$$e^x \sim \frac{1}{L}[e^L - 1] - \sum_{n=1}^{\infty}\left[\frac{n^2\pi^2}{L^2}A_n + \frac{2}{L}((-1)^n e^L - 1)\right]\cos\left(\tfrac{n\pi x}{L}\right).$$

Comparing this to the original function, we see

$$A_0 = \frac{1}{L}\left(e^L - 1\right)$$

and

$$A_n = -\frac{n^2\pi^2}{L^2} A_n + \frac{2}{L}((-1)^n e^L - 1),$$

so

$$A_n = \frac{2L}{L^2 + n^2\pi^2} \left((-1)^n e^L - 1\right), \quad n \geq 1.$$

3.4.11. (15pts) Consider the nonhomogeneous PDE:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Assume the IC $u(x,0) = f(x)$ and BC $u(0,t) = 0$ and $u(L,t) = 0$. The Dirichlet BCs imply an eigenfunction expansion of the form:

$$u(x) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}.$$

We can readily differentiate w.r.t. t and the homogeneous BCs allow two derivatives w.r.t. x, giving:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{dB_n}{dt} \sin\left(\tfrac{n\pi x}{L}\right) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 B_n(t) \sin\left(\tfrac{n\pi x}{L}\right).$$

Assume that $g(x)$ can be expanded into a sine series giving:

$$g(x) = \sum_{n=1}^{\infty} G_n \sin\left(\tfrac{n\pi x}{L}\right).$$

From the nonhomogeneous PDE, we have

$$\sum_{n=1}^{\infty} \frac{dB_n}{dt} \sin\left(\tfrac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(t) \sin\left(\tfrac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} G_n \sin\left(\tfrac{n\pi x}{L}\right).$$

Collecting the coefficients of the sine series gives the nonhomogeneous ODE:

$$\frac{dB_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 B_n(t) = G_n.$$

This equation is readily solved using techniques from linear ODEs (integrating factors) and the solution satisfies:

$$B_n(t) = A_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \left(\frac{L^2}{kn^2\pi^2}\right) G_n.$$

From the IC,

$$f(x) = \sum_{n=1}^{\infty} B_n(0) \sin\left(\tfrac{n\pi x}{L}\right),$$

so

$$B_n(0) = \frac{2}{L}\int_0^L f(x) \sin\left(\tfrac{n\pi x}{L}\right)\, dx.$$

However, we have

$$B_n(0) = A_n + \left(\frac{L^2}{kn^2\pi^2}\right) G_n,$$

which gives

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi x}{L}\right)\, dx - \left(\frac{L^2}{kn^2\pi^2}\right) G_n.$$

It follows that

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\left(\tfrac{n\pi x}{L}\right),$$

with

$$B_n(t) = \left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx - \left(\frac{L^2}{kn^2\pi^2}\right) G_n\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \left(\frac{L^2}{kn^2\pi^2}\right) G_n.$$

Written part of WeBWorK Problems. Points are exclusively for the graphs, as WeBWorK graded the Fourier coefficients.

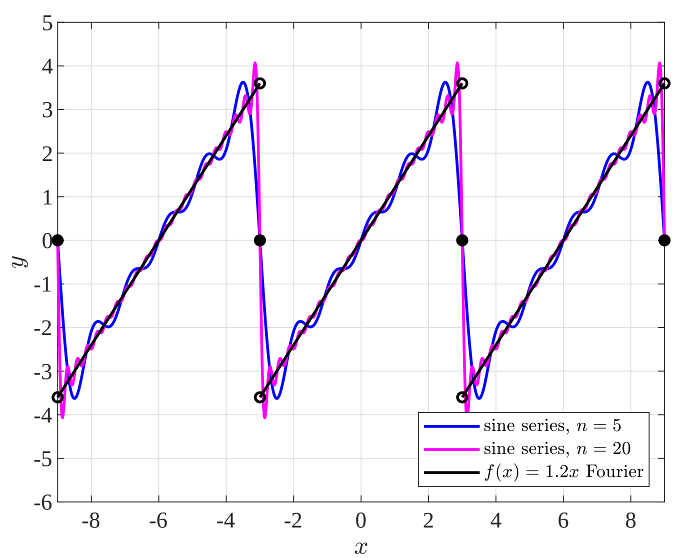
WW Fourier 3.2.2a. (5pts) Finding the Fourier series of

$$f(x) = 1.2x,$$

where the coefficient and the interval of interest varies. Since this is an odd function, $a_n = 0$ for $n = 0, 1, \ldots$ The sine series satisfies

$$f(x) \sim \sum_{n=1}^{\infty} \frac{7.2(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{3}\right).$$

Below shows in black where this Fourier series converges and the truncated series graphs for $n = 5$ and 20 terms in the series.



WW Fourier 3.2.2b. (5pts) Finding the Fourier series of

$$f(x) = 1.5e^{-x},$$

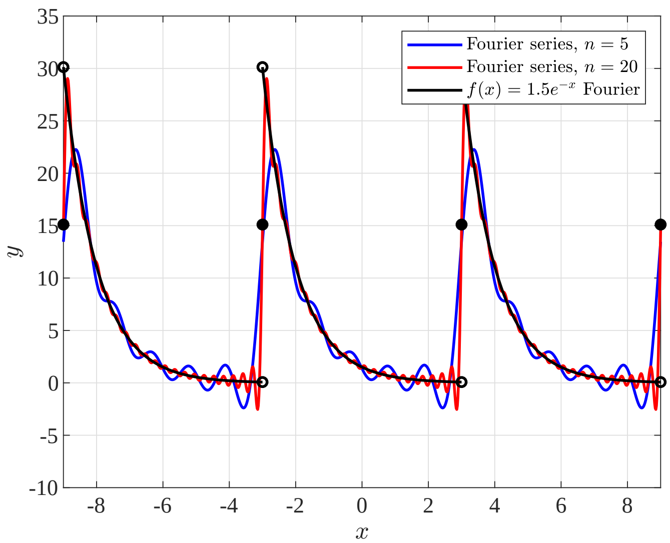
where the coefficient and the interval of interest varies. The Fourier series satisfies

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right) \right),$$

where $a_0 = \frac{e^3 - e^{-3}}{4}$, and

$$a_n = \frac{4.5(e^3 - e^{-3})(-1)^n}{9 + n^2\pi^2} \qquad \text{and} \qquad b_n = \frac{1.5n\pi(e^3 - e^{-3})(-1)^n}{9 + n^2\pi^2}.$$

Below shows in black where this Fourier series converges and the truncated series graphs for $n = 5$ and 20 terms in the series.



WW Fourier 3.3.2b. (5pts) Finding the Fourier sine series of

$$f(x) = \begin{cases} 1, & x < \frac{2}{3}, \\ 5, & \frac{2}{3} \le x < 2, \\ 0, & x > 2 \end{cases}$$

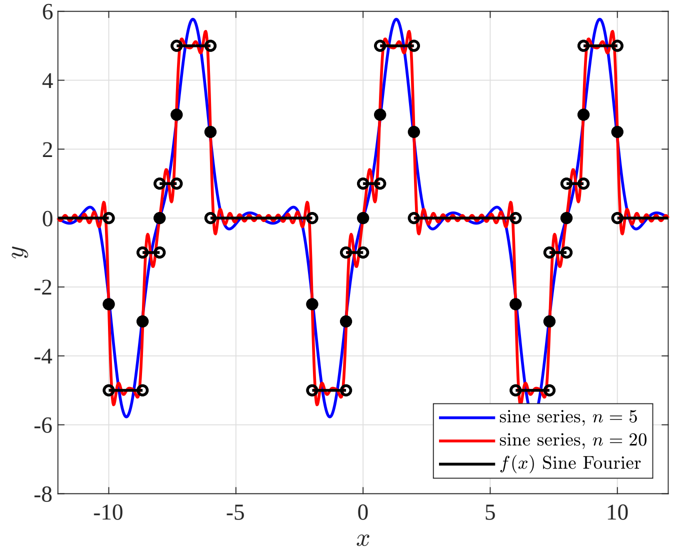
where the step values and the intervals of interest vary. The Fourier series satisfies

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right),$$

where

$$b_n = \frac{2}{n\pi}\left(\left(1 - \cos\left(\frac{n\pi}{6}\right)\right) + 5\left(\cos\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{2}\right)\right)\right).$$

Below shows in black where this Fourier series converges and the truncated series graphs for $n = 5$ and 20 terms in the series.



WW Fourier 3.3.5c. (5pts) Finding the Fourier cosine series of

$$f(x) = \begin{cases} 0, & x < 3, \\ 1.1x, & x > 3, \end{cases}$$

where the linear coefficient and the intervals of interest vary. The Fourier series satisfies

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{6}\right),$$

where $a_0 = 2.475$ and

$$a_n = \frac{6.6}{n^2\pi^2}\left(2(-1)^n - 2\cos\left(\frac{n\pi}{2}\right) - n\pi\sin\left(\frac{n\pi}{2}\right)\right).$$

Below shows in black where this Fourier series converges and the truncated series graphs for $n = 5$ and 20 terms in the series.