2.4.1 Consider the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{with BCs} \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0.$$

This is the heat equation with insulated BCs. Let $u(x,t) = \phi(x)h(t)$, then by separation of variables $\phi h' = k\phi'' h$, so

$$\frac{h'}{kh} = \frac{\phi''}{\phi} = -\lambda.$$

The resulting Sturm-Liouville problem is

$$\phi'' + \lambda\phi = 0 \quad \text{with} \quad \phi'(0) = 0 \quad \text{and} \quad \phi'(L) = 0.$$

This problem can be shown to have eigenvalue $\lambda_0 = 0$ with eigenfunction $\phi_0(x) = 1$ and eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ with eigenfunctions $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$. The time varying problem has $h_n(t) = c_n e^{-kn^2\pi^2 t/L^2}$. This gives

$$u_n(x,t) = A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, ...$$

The superposition principle gives

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

We demonstrated that there were no eigenvalues $\lambda < 0$. The Fourier coefficients are

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

a. (8pts)Consider the initial condition:

$$f(x) = u(x,0) = \begin{cases} 0 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases}$$

The Fourier coefficients are:

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{L/2}^L dx = \frac{1}{L}\left[L - \frac{L}{2}\right] = \frac{1}{2}$$

and

$$A_n = \frac{2}{L}\int_{L/2}^L \cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^L = \frac{2}{n\pi}\left[\sin(n\pi) - \sin(\frac{n\pi}{2})\right] = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

The superposition principle gives the solution:

$$u(x,t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2}{L^2}kt} \cos\left(\frac{n\pi x}{L}\right).$$

c) (7pts) Consider the initial condition:

$$f(x) = u(x,0) = -2\sin\frac{\pi x}{L}.$$

The Fourier coefficients are:

$$A_0 = \frac{1}{L}\int_0^L \left(-2\sin\left(\frac{\pi x}{L}\right)\right)\,dx = \frac{2}{\pi}\cos\left(\frac{\pi x}{L}\right)\Big|_0^L = \frac{2}{\pi}[\cos(\pi) - \cos(0)] = -\frac{4}{\pi}$$

and

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L -2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = -\frac{4}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= -\frac{2}{L}\int_0^L \sin\left(\frac{\pi}{L}(1+n)x\right) + \sin\left(\frac{\pi}{L}(1-n)x\right)\,dx. \end{aligned}$$

For $n = 1$,

$$A_1 = -\frac{2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\,dx = \frac{1}{\pi}\cos\left(\frac{2\pi x}{L}\right)\Big|_0^L = 0.$$

For $n > 1$,

$$\begin{aligned} A_n &= \frac{2}{\pi}\left[\frac{1}{(1+n)}\cos\frac{\pi}{L}(1+n)x \;+\; \frac{1}{(1-n)}\cos\frac{\pi}{L}(1-n)x\right]_0^L \\ &= \frac{2}{\pi}\left[\frac{1}{(1+n)}\left(\cos(\pi(1+n)) - 1\right) + \frac{1}{(1-n)}\left(\cos(\pi(1-n)) - 1\right)\right] \\ &= \frac{2}{\pi}\left[\frac{1}{(1+n)}\left((-1)^{1+n} - 1\right) \;+\; \frac{1}{(1-n)}\left((-1)^{1-n} - 1\right)\right]. \end{aligned}$$

Thus, if n is odd, then $A_n = 0$. If n is even, then

$$A_n = \frac{2}{\pi}\left[\frac{-2}{1+n} + \frac{-2}{1-n}\right] = \frac{8}{\pi(n^2-1)}.$$

All even numbers are of the form $A_{2n} = \frac{8}{\pi(4n^2-1)}$, so it follows that

$$u(x,t) = -\frac{4}{\pi} + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}e^{\frac{-4n^2\pi^2}{L^2}kt}\cos\left(\frac{2n\pi x}{L}\right).$$

2.4.2 (10pts) Consider the PDE:

$$u_t = u_{xx}, \qquad \frac{\partial u}{\partial x}(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = f(x).$$

Use separation of variables $u(x,t) = \phi(x)G(t)$, so

$$\phi(x)G'(t) = k\phi''(x)G(t) \qquad \text{or} \qquad \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

This gives the ODE in t as $\frac{dG}{dt} = -k\lambda G$, which has the solution:

$$G(t) = ce^{-\lambda kt}.$$

The Sturm-Liouville problem is

$$\phi'' + \lambda\phi = 0 \qquad \text{with BCs} \quad \phi'(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Case i: Suppose $\lambda = -\alpha^2 < 0$. This solution will grow in time because of the form of $G(t)$, contradicting one of the assumptions. Alternately,

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

Since $\phi'(0) = 0$, $c_2 = 0$. The other BC gives $\phi(L) = c_1 \cosh(\alpha L) = 0$, so $c_1 = 0$, leaving only the trivial solution.

Case ii: Suppose that $\lambda = 0$, then $\phi(x) = c_1 x + c_2$. The first BC $\phi'(0) = c_1 = 0$. The other BC gives $\phi(L) = c_2 = 0$, so again there is only the trivial solution.

Case iii: Suppose that $\lambda = \alpha^2 > 0$, then

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Since $\phi'(0) = 0$, $c_2 = 0$. The other BC gives

$$\phi(L) = c_1 \cos(\alpha L) = 0 \qquad \text{or} \qquad \alpha L = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, ...$$

It follows that we have eigenvalues and eigenfunctions:

$$\lambda_n = \frac{((2n-1)\pi)^2}{4L^2} \qquad \text{and} \qquad \phi_n(x) = \cos\left(\frac{(2n-1)\,\pi x}{2L}\right), \quad n = 1, 2, ...$$

By the superposition principle, the solution is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{((2n-1)\pi)^2}{4L^2}t} \cos\left(\frac{(2n-1)\,\pi x}{2L}\right).$$

The initial condition is

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\,\pi x}{2L}.$$

Multiply by the m^{th} eigenfunction and integrate 0 to L, so

$$\begin{aligned} \int_0^L f(x) \cos\frac{(2m-1)\,\pi x}{2L}\,dx &= \sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{(2n-1)\,\pi x}{2L} \cos\frac{(2m-1)\,\pi x}{2L}\,dx \\ &= A_m \left(\frac{L}{2}\right), \end{aligned}$$

from the orthogonality of the eigenfunctions (or computing the integrals $m \neq n$). It follows that the Fourier coefficients are:

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\frac{(2n-1)\,\pi x}{2L}\,dx.$$

2.5.1 (10pts) d. Consider the Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The BCs are:

$$u(0,y) = g(y), \qquad u(L,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad u(x,H) = 0.$$

Assume $u(x,y) = h(x)\phi(y)$, so

$$h''(x)\phi(y) + h(x)\phi''(y) = 0 \qquad \text{or} \qquad \frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \lambda.$$

This leads to an ODE in x:

$$h''(x) - \lambda h(x) = 0, \qquad h(L) = 0.$$

and the Sturm-Liouville problem in y

$$\phi''(y) + \lambda\phi(y) = 0, \qquad \phi'(0) = 0, \qquad \phi(H) = 0.$$

Consider 3 cases: i. $\lambda = -\alpha^2 < 0$, so $\phi(y) = c_1\cosh(\alpha y) + c_2\sinh(\alpha y)$. Since $\phi'(0) = 0$, we have $c_2 = 0$. Then $\phi(H) = c_1\cosh(\alpha H) = 0$ implies $c_1 = 0$, so only the trivial solution solves this case.
ii. $\lambda = 0$, so $\phi(y) = c_1 y + c_2$. Since $\phi'(0) = 0$, we have $c_1 = 0$. Then $\phi(H) = c_2 = 0$ implies $c_2 = 0$, so only the trivial solution solves this case.
iii. $\lambda = \alpha^2 > 0$, so $\phi(y) = c_1\cos(\alpha y) + c_2\sin(\alpha y)$. Since $\phi'(0) = 0$, we have $c_2 = 0$. Then $\phi(H) = c_1\cos(\alpha H) = 0$ implies we have eigenvalues and eigenfunctions:

$$\lambda_n = \frac{(n-1/2)^2\pi^2}{H^2}, \qquad \phi_n = \cos\left(\frac{(n-1/2)\pi y}{H}\right), \qquad n = 1,2,...$$

The first ODE becomes, $h''(x) - \lambda_n h(x) = 0$, or

$$h(x) = c_1\cosh\left(\frac{(n-1/2)\pi}{H}(L-x)\right) + c_2\sinh\left(\frac{(n-1/2)\pi}{H}(L-x)\right).$$

Since $h(L) = 0$, it is easy to see that

$$h(L) = c_1\cosh(0) = 0 \qquad \text{or} \qquad c_1 = 0.$$

Thus,

$$h(x) = c_2\sinh\left(\frac{(n-1/2)\pi}{H}(L-x)\right).$$

The product solution is

$$u_n(x,y) = A_n\cos\left(\frac{(n-1/2)\pi y}{H}\right)\sinh\left(\frac{(n-1/2)\pi}{H}(L-x)\right), \qquad n = 1,2,...$$

Thus the principle of superposition leads to the general solution:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(n-1/2)\pi y}{H}\right) \sinh\left(\frac{(n-1/2)\pi}{H}(L-x)\right)$$

The left edge BC gives:

$$u(0,y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{(n+1/2)\pi y}{H}\right) \sinh\left(\frac{(n-1/2)\pi L}{H}\right) = g(y).$$

Multiplying by $\phi_m(y)$ and integrating $y \in [0,H]$, we obtain:

$$\int_0^H g(y)\phi_m(y)\,dy = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{(n-1/2)\pi L}{H}\right) \int_0^H \phi_n(y)\phi_m(y)\,dy.$$

By orthogonality,

$$\int_0^H \phi_n(y)\phi_m(y)\,dy = \left\{ \begin{array}{ll} 0, & m \neq n, \\ H/2, & m = n \end{array} \right.,$$

we obtain the Fourier coefficients:

$$A_n = \frac{2}{H \sinh\left(\frac{(n-1/2)\pi L}{H}\right)} \int_0^H g(y) \cos\left(\frac{(n-1/2)\pi y}{H}\right) dy \quad n = 1, 2, ...$$

2.5.1 (10pts) g. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The BCs are

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = \left\{ \begin{array}{ll} 0 & x > \frac{L}{2} \\ 1 & x < \frac{L}{2} \end{array} \right., \qquad \frac{\partial u}{\partial y}(x,H) = 0.$$

Assume $u(x,y) = \phi(x)G(y)$, and we obtain the two ODEs as before,

$$\begin{array}{rcl} \phi'' + \lambda\phi(x) & = & 0, \quad \phi'(0) = 0, \quad \phi'(L) = 0 \\ G''(y) - \lambda G(y) & = & 0, \quad G'(H) = 0 \end{array}$$

The first ODE is a SL problem, which we have solved before. It has eigenvalues and eigenfunctions: $\lambda_0 = 0$ with $\phi_0(x) = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $\phi_n = \cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, ...$
The second ODE has the solution:

$$G_n(y) = c_1 \cosh\left(\frac{n\pi}{L}(H-y)\right) + c_2 \sinh\left(\frac{n\pi}{L}(H-y)\right).$$

The insulated BC $G_n'(H) = 0$ gives:

$$G_n'(y) = -c_1 \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L}(H-y)\right) - c_2 \frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}(H-y)\right),$$

so $G_n'(H) = -c_2\frac{n\pi}{L} = 0$ or $c_2 = 0$. Thus, $G_n(y) = c_1 \cosh\left(\frac{n\pi}{L}(H-y)\right)$. The special case, $\lambda_0 = 0$ with $G_0'(H) = 0$, gives the solution $G_0(y) = 1$.
The superposition principle gives:

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi}{L}(H-y)\right).$$

The bottom BC gives

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi}{L}(H)\right) = f(x) = \begin{cases} 0, & x > \frac{L}{2} \\ 1, & x < \frac{L}{2}. \end{cases}$$

Once again we use orthogonality to show that the Fourier coefficients are

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{L/2} 1\,dx = \frac{1}{2}.$$

and

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi H}{L}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx,$$

or

$$\begin{aligned} A_n &= \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)}\frac{L}{n\pi}\sin\left(\frac{n\pi L}{2L}\right) \\ &= \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi\cosh\left(\frac{n\pi H}{L}\right)} \end{aligned}$$

The solution becomes:

$$u(x,y = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi\cosh\left(\frac{n\pi H}{L}\right)}\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi}{L}(H-y)\right).$$

2.5.2 Consider Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < L, \qquad 0 < y < H,$$

with the flux boundary conditions:

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,H) = f(x).$$

a. (2pts) The solvability conditions says that the net heat flow through the boundary must be zero in order for a steady state to exist. As 3 sides of the rectangle are insulated, the net flow through the last side must also be zero, so $\int_0^L f(x)\,dx = 0$.

b. (10pts) Assume $u(x,y) = \phi(x)G(y)$, and as before, we get the two ODEs

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \qquad \phi'(0) = 0, \qquad \phi'(L) = 0 \\ G''(y) - \lambda G(y) &= 0, \qquad G'(0) = 0. \end{aligned}$$

For the first ODE is the SL problem and has been solved before. It has eigenvalues and eigenfunctions: $\lambda_0 = 0$ with $\phi_0(x) = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with $\phi_n = \cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, ...$

For the second problem, we get the solution:

$$G_n(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right).$$

It is easy to see that the BC $G_n'(0) = 0$ implies that $c_2 = 0$. Thus, we have:

$$G_n(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right).$$

The special case, $\lambda_0 = 0$ with $G_0'(0) = 0$, gives the solution $G_0(y) = 1$.
The superposition principle gives the general solution:

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right).$$

We check the flux BC at the top.

$$\frac{\partial u}{\partial y}(x, H) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right) = f(x).$$

The solvability condition from Part a states:

$$\int_0^L f(x)\ dx = 0 = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right) \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx,$$

which holds because $\int_0^L \cos\left(\frac{n\pi x}{L}\right) dx = 0$.
We use orthogonality to find the Fourier coefficients:

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right)\ dx = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right) \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx.$$

From the orthogonality of the cosine functions:

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

c. (3pts) Consider the time-dependent heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$, where the IC is $u(x, y, 0) = g(x, y)$. We integrate over the region and see that the rate of change of heat energy is

$$\int_0^H \int_0^L \frac{\partial u}{\partial t}\, dx\, dy = k \int_0^H \int_0^L \nabla^2 u\, dx\, dy = k \left[\int_0^H \int_0^L \frac{\partial^2 u}{\partial x^2}\, dx\, dy + \int_0^H \int_0^L \frac{\partial^2 u}{\partial y^2}\, dx\, dy\right].$$

Integrating the second partials give the conditions on the boundaries, which all integrate to zero, so

$$k \left[\int_0^H \left(\frac{\partial u}{\partial x}(L, y) - \frac{\partial u}{\partial x}(0, y)\right) dy + \int_0^L \left(\frac{\partial u}{\partial y}(x, H) - \frac{\partial u}{\partial y}(x, 0)\right) dx\right] = k \int_0^L f(x)\, dx = 0.$$

It follows that the rate of change of heat is zero, so the heat in the region is conserved or constant. That is:

$$\int_0^H \int_0^L u(x,y,t)\,dx\,dy = C.$$

So the steady-state temperature distribution contains the same heat as the initial distribution:

$$\int_0^H \int_0^L \lim_{t\to\infty} u(x,y,t)\,dx\,dy = \int_0^H \int_0^L u(x,y)\,dx\,dy = \int_0^H \int_0^L u(x,y,0)\,dx\,dy = C.$$

However,

$$\begin{aligned}
\int_0^H \int_0^L u(x,y)\,dx\,dy &= \int_0^H \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)\right) dx\,dy,\\
&= \int_0^H \int_0^L A_0\,dx\,dy + \int_0^H \left(A_n \cosh\left(\frac{n\pi y}{L}\right)\left[\sin\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\right]_0^L\right) dy,\\
&= \int_0^H \int_0^L A_0\,dx\,dy \;=\; HLA_0.
\end{aligned}$$

It follows that

$$A_0 = \frac{1}{HL}\int_0^H \int_0^L u(x,y,0)\,dx\,dy = \frac{1}{HL}\int_0^H \int_0^L g(x,y)\,dx\,dy,$$

the average of the IC.

2.5.6 (8pts) b. Consider Laplace's equation:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < a, \qquad 0 < \theta < \pi.$$

with the BCs (insulated on the x-axis):

$$\frac{\partial u}{\partial r}(r,0) = 0, \qquad \frac{\partial u}{\partial r}(r,\pi) = 0, \qquad 0 \le r \le a, \qquad u(a,\theta) = g(\theta), \qquad 0 \le \theta \le \pi.$$

Assume $u(r,\theta) = \phi(\theta)G(r)$, then separation of variables gives:

$$\frac{\theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{1}{r^2}\phi'' G = 0 \qquad \text{or} \qquad -\frac{\phi''}{\phi} = \frac{r}{G}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = \lambda.$$

This gives the Sturm-Liouville problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi'(0) = 0, \qquad \phi'(\pi) = 0,$$

and the ODE in r

$$rG'' + G' - \frac{1}{r}\lambda G = 0.$$

As we have seen before, the Sturm-Liouville problem has eigenvalues and eigenfunctions:

$$\lambda_0 = 0 \quad \text{with} \quad \phi_0(\theta) = 1,$$

and

$$\lambda_n = n^2 \quad \text{with} \quad \phi_n(\theta) = \cos(n\theta), \quad n = 1, 2, ...$$

For $\lambda_0 = 0$, the solution to the equation, $r^2 G'' + rG' = 0$ is

$$G_0(r) = c_1 + c_2 \ln(r),$$

while for $\lambda_n = n^2$, we have shown the solution to be:

$$G_n(r) = c_1 r^n + c_2 r^{-n}.$$

In both cases, the boundedness at $r = 0$ implies that $c_2 = 0$.
The superposition principle gives the general solution:

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta).$$

The BC gives:

$$u(a, \theta) = a_0 + \sum_{n=1}^{\infty} a_n a^n \cos(n\theta) = g(\theta).$$

Using orthogonality, we obtain the Fourier coefficients:

$$a_0 = \frac{1}{\pi} \int_0^{\pi} g(\theta)\, d\theta,$$

and

$$a_n = \frac{2}{\pi a^n} \int_0^{\pi} g(\theta) \cos(n\theta)\ d\theta.$$

2.5.8 (12pts) b. Consider Laplace's equation on an annular region:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad a < r < b, \qquad -\pi < \theta < \pi.$$

with the BCs, including implicit ones,

$$\frac{\partial u}{\partial r}(a, \theta) = 0, \qquad u(b, \theta) = g(\theta), \qquad u(r, -\pi) = u(r, \pi), \qquad \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi).$$

Assume $u(r, \theta) = \phi(\theta)G(r)$, then separation of variables gives:

$$\frac{r}{G}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

In θ, we obtain the periodic Sturm-Liouville problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi(-\pi) = \phi(\pi), \qquad \phi'(-\pi) = \phi'(\pi).$$

The second ODE is Cauchy's equation:

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0, \qquad G'(a) = 0.$$

As we have seen before, the Sturm-Liouville problem has eigenvalues and eigenfunctions:

$$\lambda_0 = 0 \quad \text{with} \quad \phi_0(\theta) = 1,$$

and

$$\lambda_n = n^2 \quad \text{with} \quad \phi_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta), \quad n = 1, 2, ...$$

For $\lambda_0 = 0$, the solution to the equation, $r^2 G'' + rG' = 0$ is

$$G_0(r) = c_1 + c_2 \ln(r),$$

while for $\lambda_n = n^2$, we have shown the solution to Cauchy's equation to be:

$$G_n(r) = c_1 r^n + c_2 r^{-n}.$$

From the boundary condition, $G'(a) = 0$, we see

$$G_0'(r) = \frac{c_2}{r}, \qquad \text{so} \quad G_0'(a) = \frac{c_2}{a} = 0.$$

Thus, $c_2 = 0$. Also,

$$G_n'(r) = nc_1 r^{n-1} - nc_2 r^{-n-1}, \qquad \text{so} \quad G_n'(a) = nc_1 a^{n-1} - nc_2 a^{-n-1} = 0.$$

It follows that $c_2 = c_1 a^{2n}$ for $n = 1, 2, ...$, so

$$G(r) = c_1 \left(r^n + a^{2n} r^{-n} \right).$$

The superposition principle gives the general solution:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left(r^n + a^{2n} r^{-n} \right) \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right).$$

The BC at $r = b$ gives

$$u(b, \theta) = A_0 + \sum_{n=1}^{\infty} \left(b^n + a^{2n} b^{-n} \right) \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right) = g(\theta).$$

We use orthogonality to obtain the Fourier coefficients:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta,$$

and

$$A_n \left(b^n + a^{2n} b^{-n} \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) \, d\theta,$$

and

$$B_n \left(b^n + a^{2n} b^{-n} \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) \, d\theta.$$

The solution can be written:

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta \\ &+ \sum_{n=1}^{\infty} \left(\frac{r^n + a^{2n} r^{-n}}{b^n + a^{2n} b^{-n}} \right) \frac{1}{\pi} \left[\int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta \cdot \cos(n\theta) + \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta \cdot \sin(n\theta) \right]. \end{aligned}$$

2.5.10 (5pts) Consider Poisson's equation $\nabla^2 u = g(x)$ with the boundary condition, $u = f(x)$. Assume the solution is not unique. Then there is a u and v, which both satisfy the Poisson's equation and the boundary conditions, *i.e.,*

$$\nabla^2 u = g(x), \quad \text{with} \quad u = f(x), \qquad \text{and} \qquad \nabla^2 v = g(x), \quad \text{with} \quad v = f(x).$$

Define $w = u - v$, so $\nabla^2 w = \nabla^2 u - \nabla^2 v = g(x) - g(x) = 0$. Similarly, $w = u - v = f(x) - f(x) = 0$ on the boundary. It follows that w is a solution of the Laplace equation. The maximum principle then gives $0 \le w \le 0$ or $w = 0$, which implies $u = v$. Therefore the solution is unique.

2.5.15 (10pts) b. Consider Laplace's equation on a semi-infinite strip:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \infty, \qquad 0 < y < H,$$

with the BCs:

$$u(x,0) = 0, \qquad u(x,H) = 0, \qquad u(0,y) = f(y), \qquad u(x,y) \text{ bounded for } x \to \infty.$$

Assume $u(x,y) = h(x)\phi(y)$, then separation of variables gives the SL problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0, \qquad \phi(H) = 0,$$

and the ODE in x:

$$h'' - \lambda h = 0.$$

We have often solved this SL problem, and we obtain the eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad \text{and} \qquad \phi_n = \sin\left(\frac{n\pi y}{H}\right), \qquad n = 1, 2, ...$$

The general solution for the second problem is

$$h(x) = c_1 e^{\left(\frac{n\pi x}{H}\right)} + c_2 e^{\left(-\frac{n\pi x}{H}\right)}.$$

Since this solution must be bounded, $c_1 = 0$, hence,

$$h(x) = c_2 e^{-\left(\frac{n\pi x}{H}\right)}$$

The superposition principle gives the general solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) e^{-\left(\frac{n\pi x}{H}\right)}.$$

The left BC gives

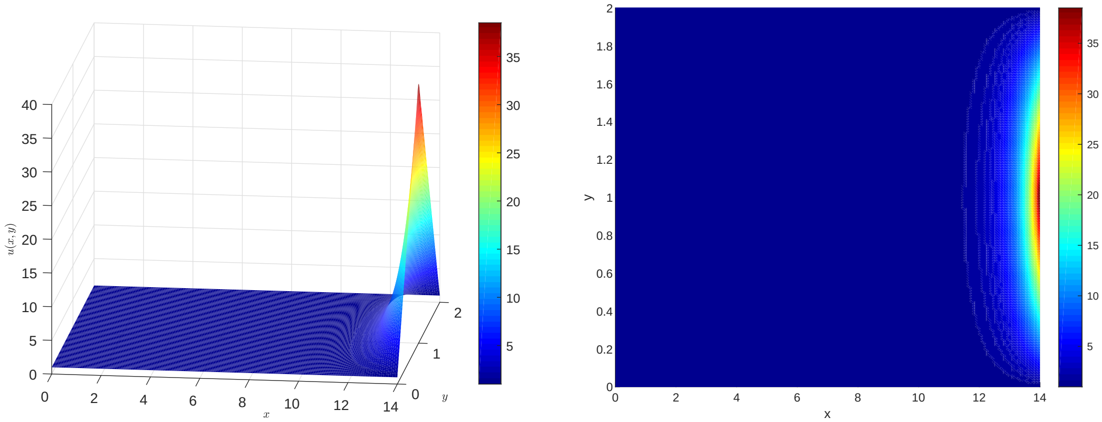
$$u(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) = f(y).$$

The Fourier coefficients are readily found with

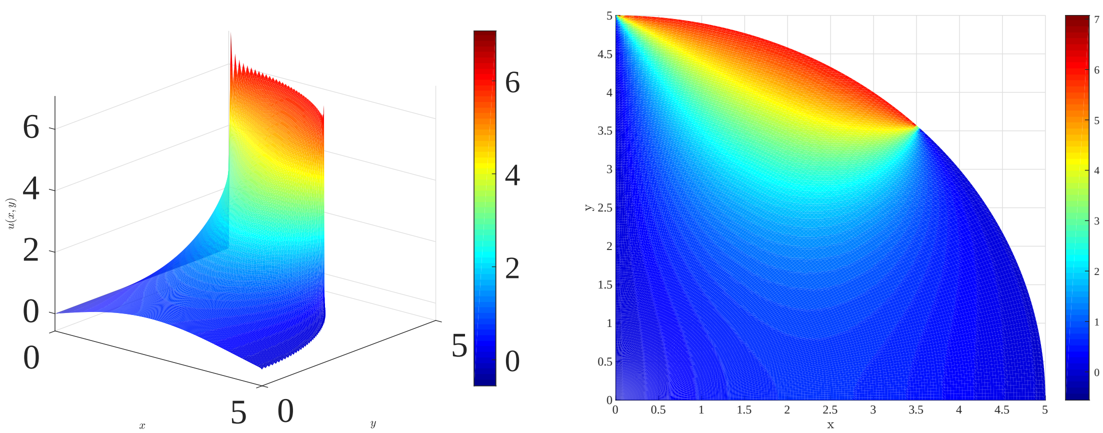
$$A_n = \frac{2}{H}\int_0^H f(y) \sin\left(\frac{n\pi y}{H}\right) dy.$$

Bonus: Graphs for WeBWorK Problems 5 and 6. Students only need to include one appropriate surface graph for either problem, and the numbers on the graphs will vary based on different numbers from WeBWorK. However the basic shapes should be the same.

WeBWorK Problem 5. (7pts) Laplace's equation with zero (Dirichlet) boundary conditions along the y-edges, insulated at $x = 0$, and a tent function at $x = L$ (where L varies and height of tent varies). Below shows a 3D surface and the heat map in the plane.



WeBWorK Problem 6. (8pts) Laplace's equation with zero (Dirichlet) boundary conditions along the vertical axis, insulated along $x = 0$, and a step function at $r = a$ (where a varies and height of the step varies). Below shows a 3D surface and the heat map in the plane.



Below is the MatLab code for generating the figures above for WW Problem 6. The key to creating the polar surface is line 33, using the MatLab function *pol2cart*.

```
1  format compact;
2  a = 5;
3  H = pi/2;
4  NptsR=151;             % number of x pts
5  NptsTH=151;             % number of t pts
6  Nf=150;                 % number of Fourier terms
7  r=linspace(0,a,NptsR);
```

```
t=linspace(0,H,NptsTH);
[R,TH]=ndgrid(r,t);

fs=8;
figure(1)
clf

b=zeros(1,Nf);
U=zeros(NptsTH,NptsR);

for n=1:Nf
    if (mod(n,4)==1)
       b(n)=2.23754/((2*n-1)*5^(2*n-1));  % Fourier coefficients
    elseif (mod(n,4)==2)
       b(n)=-13.0413/((2*n-1)*5^(2*n-1));  % Fourier coefficients
    elseif (mod(n,4)==3)
       b(n)=13.0413/((2*n-1)*5^(2*n-1));  % Fourier coefficients
    else
       b(n)=-2.23754/((2*n-1)*5^(2*n-1));  % Fourier coefficients
    end
    Un=b(n)*(R.^(2*n-1)).*cos((2*n-1)*TH);  % Temperature(n)
    U=U+Un;
end

set(gca,'FontSize',[fs]);
[R,TH] = pol2cart(TH,R);
surf(R,TH,U);
shading interp
colormap(jet)
fontlabs = 'Times New Roman';
xlabel('$x$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
ylabel('$y$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
zlabel('$u(x,y)$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
%axis tight
colorbar
view([40 20])
print -depsc WW6_a.eps

figure(2)
clf

set(gca,'FontSize',[fs]);
surf(R,TH,U);
shading interp
colormap(jet)
view([0 90])               %create 2D color map of temperature
xlabel('x','Fontsize',fs); ylabel('y','Fontsize',fs); ...
    zlabel('u(x,y)','Fontsize',fs);axis tight
colorbar
set(gca,'FontSize',[fs]);

print -depsc WW6_b.eps
```