1.5.22 (5pts) The text gives us (1.5.25) as:

$$\nabla^2 T = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial T}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial T}{\partial w}\right)\right],$$

where if $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$, and $\mathbf{r} \equiv x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. If $h_u = |\partial\mathbf{r}/\partial u|$, etc., then in cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $u = r,\ v = \theta,\ w = z$.

$$\mathbf{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} \qquad \frac{\partial\mathbf{r}}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \frac{\partial\mathbf{r}}{\partial\theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \qquad \frac{\partial\mathbf{r}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} h_r &= \sqrt{\cos^2\theta + \sin^2\theta + 0^2} = 1 \\ h_\theta &= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta + 0^2} = \sqrt{r^2} = r \\ h_z &= \sqrt{1^2} = 1 \end{aligned}$$

Thus, we have

$$\begin{aligned} \nabla^2 T &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial T}{\partial z}\right)\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{\partial^2 T}{\partial z^2} \end{aligned}$$

2.2.4 a. (2pts) Assume that L is a linear operator, and suppose $L(u) = f$. If u_p is a particular solution with $L(u_p) = f$ and if u_1 and u_2 are homogeneous solutions, so $L(u_1) = L(u_2) = 0$. Then consider

$$L(u_p + c_1u_1 + c_2u_2) = L(u_p) + c_1L(u_1) + c_2L(u_2) = f + 0 + 0 = f.$$

which implies that $u_p + c_1u_1 + c_2u_2$ is another particular solution.

b. (2pts) Assume that u_{p_1} and u_{p_2} are particular solutions, such that $L(u_{p_1}) = f_1$ and $L(u_{p_2}) = f_2$. Let $u = u_{p_1} + u_{p_2}$, then

$$L(u) = L(u_{p_1} + u_{p_2}) = L(u_{p_1}) + L(u_{p_2}) = f_1 + f_2.$$

It follows that $u = u_{p_1} + u_{p_2}$ is a particular solution for $f_1 + f_2$.

2.3.1.b. (4pts) Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}.$$

Let $u(x,t) = \phi(x)G(t)$, then

$$\phi(x)\frac{dG(t)}{dt} = k\frac{d^2\phi}{dx^2}G(t) - v_0\frac{d\phi}{dx}G(t).$$

Dividing this equation by $\phi(x)G(t)$ gives

$$\frac{1}{G}\frac{dG(t)}{dt} = \frac{1}{\phi}\left(k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx}\right) = -\lambda,$$

where the left hand side depends on t and the right hand side depends on x, so is constant. This yields two ODEs

$$\frac{dG}{dt} = -\lambda G, \qquad \text{and} \qquad k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx} = -\lambda\phi.$$

c. (4pts) Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Let $u(x,y) = \phi(x)G(y)$, then

$$\frac{d^2\phi}{dx^2}G + \phi\frac{d^2G}{dy^2} = 0.$$

Dividing this equation by $\phi(x)G(y)$ and rearranging gives

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -\lambda,$$

where the left hand side depends on x and the right hand side depends on y, so is constant. This yields two ODEs

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \text{and} \qquad \frac{d^2G}{dy^2} - \lambda G = 0.$$

f. (4pts) Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$

Let $u(x,t) = \phi(x)G(t)$, then

$$\phi\frac{d^2G}{dt^2} = c^2\frac{d^2\phi}{dx^2}G.$$

Dividing this equation by $c^2\phi(x)G(t)$ gives

$$\frac{1}{c^2G}\frac{d^2G}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda,$$

where the left hand side depends on t and the right hand side depends on x, so is constant. This yields two ODEs

$$\frac{d^2G}{dt^2} + c^2\lambda G = 0, \qquad \text{and} \qquad \frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

2.3.8 Consider the PDE:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u, \qquad \text{with BCs} \quad u(0,t) = 0 \quad \text{and} \quad u(L,t), 0.$$

a. (5pts) The equation for equilibrium $u_e(x)$ is

$$k\frac{d^2 u_e}{dx^2} - \alpha u_e = 0, \qquad \text{with} \quad u_e(0) = 0 \quad \text{and} \quad u_e(L), 0.$$

The characteristic equation is $\lambda^2 - \frac{\alpha}{k} = 0$, so the general solution ($\alpha, k > 0$) is

$$u_e(x) = c_1 \cosh\left(\sqrt{\frac{\alpha}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{\alpha}{k}}x\right).$$

The BC $u_e(0) = c_1 = 0$. The other BC $u_e(L) = c_2 \sinh\left(\sqrt{\frac{\alpha}{k}}L\right) = 0$ implies $c_2 = 0$. It follows that the equilibrium solution is

$$u_e(x) \equiv 0.$$

b. (10pts) Assume an initial condition $u(x,0) = f(x)$. We use separation of variables $u(x.t) = \phi(x)G(t)$, then the PDE becomes

$$\phi\frac{dG(t)}{dt} = k\frac{d^2\phi}{dx^2}G(t) - \alpha\phi(x)G(t).$$

We divide by $kG\phi$ and put the term with α on the left side, then

$$\frac{1}{kG}\frac{dG(t)}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

This results in the two ODEs

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \text{and} \qquad \frac{dG}{dt} = (-\lambda k - \alpha)\, G.$$

The solution of the t equation is:

$$G(t) = c_1 e^{-(\lambda k + \alpha)t}.$$

The Sturm-Liouville problem is

$$\phi'' + \lambda\phi = 0 \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

We have previously shown that this problem only has nontrivial solutions for $\lambda > 0$. We obtain the eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{with} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, ...$$

By superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n kt} e^{-\alpha t} = e^{-\alpha t}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 kt}{L^2}}.$$

Initial conditions provide

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

We use the orthogonality of the eigenfunction to obtain the Fourier coefficients:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

For large time, $\lim_{t\to\infty} e^{-\alpha t} = 0$ and $\lim_{t\to\infty} e^{-\frac{n^2\pi^2 k}{L^2}t} = 0$.
Thus $u(x,t) \to 0$ ($u(x,t)$ will converge to the only equilibrium 0 we found in (a)).

WeBWorK Problems

5. (3pts) This one-dimensional rod has heat generated the entire length of the rod along with flux conditions at each boundary. In order to have an equilibrium, the flux at the right end needs β to be chosen such that it equal the value at the left end minus the integral of the heat source generated throughout the rod. The heat generated in the rod is the constant in the equation times the length of the rod.

6. (3pts) This one-dimensional rod has no heat sources in the rod. For there to be an equilibrium, we simply need the fluxes at each end to be the same, so that the heat entering on the left matches the heat leaving on the right.

7. (3pts) This one-dimensional rod has both ends insulated. For an equilibrium to occur, then the net amount of heat generated in the rod must be zero. It follows that β must be selected so that the integral of the nonhomogeneous function (the term following the heat diffusion term) over the length of the rod must be zero. This prevents any build up of heat in the rod over time.