1. (5pts) a. With the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

the equilibrium problem satisfies:

$$u_e'' = 0, \qquad u_e(0) = A, \quad \text{and} \quad u_e(10) = B.$$

Integrating twice gives $u_e(x) = c_1x + c_2$. The BC $u_e(0) = c_2 = A$, while $u_e(10) = 10c_1 + A = B$ or $c_1 = \frac{B-A}{10}$. It follows that

$$u_e(x) = \frac{(B-A)x}{10} + A, \qquad \text{so} \quad v(x,t) = u(x,t) - \left(\frac{(B-A)x}{10} + A\right).$$

b. From above, we clearly have $v_t = u_t$ and $v_{xx} = u_{xx}$, so $v(x,t)$ satisfies the same PDE as $u(x,t)$. Considering the BCs, we see that

$$v(0,t) = u(0,t) - u_e(0) = A - A = 0.$$

Similarly,

$$v(10,t) = u(10,t) - u_e(10) = B - B = 0,$$

so $v(x,t)$ satisfies the PDE with homogeneous boundary conditions. The initial condition for this PDE in $v(x,t)$ is given by

$$v(x,0) = u(x,0) - u_e(x) = f(x) - \left(\frac{(B-A)x}{10} + A\right).$$

2. (6pts) We show that the Laplacian in Cartesian and Polar coordinates satisfy:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We have:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Differntiating, we have:

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta).$$

From the chain rule, we have:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial u}{\partial y}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin(\theta)\frac{\partial u}{\partial x} + r\cos(\theta)\frac{\partial u}{\partial y}.$$

We use the chain and product rule to compute u_{rr} and $u_{\theta\theta}$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} = &= \cos(\theta)\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\
&= \cos(\theta)\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial r}\right) + \sin(\theta)\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) \\
&= \cos^2(\theta)\frac{\partial^2 u}{\partial x^2} + 2\cos(\theta)\sin(\theta)\frac{\partial^2 u}{\partial x \partial y} + \sin^2(\theta)\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} = &= -r\cos(\theta)\frac{\partial u}{\partial x} - r\sin(\theta)\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} - r\sin(\theta)\frac{\partial u}{\partial y} + r\cos(\theta)\frac{\partial}{\partial \theta}\frac{\partial u}{\partial y} \\
&= -r\cos(\theta)\frac{\partial u}{\partial x} - r\sin(\theta)\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right) - r\sin(\theta)\frac{\partial u}{\partial y} \\
&\quad + r\cos(\theta)\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= -r\left(\cos(\theta)\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial u}{\partial y}\right) \\
&\quad + r^2\left(\sin^2(\theta)\frac{\partial^2 u}{\partial x^2} - 2\cos(\theta)\sin(\theta)\frac{\partial^2 u}{\partial x \partial y} + \cos^2(\theta)\frac{\partial^2 u}{\partial y^2}\right) \\
&= -r\frac{\partial u}{\partial r} + r^2\left(\sin^2(\theta)\frac{\partial^2 u}{\partial x^2} - 2\cos(\theta)\sin(\theta)\frac{\partial^2 u}{\partial x \partial y} + \cos^2(\theta)\frac{\partial^2 u}{\partial y^2}\right).
\end{aligned}$$

Next we add $u_{rr} + u_{\theta\theta}/r^2$ and $\cos^2(\theta) + \sin^2(\theta) = 1$, giving:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

It follows that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

3. (5pts) Let $u(r,t) = G(t)\phi(r)$, then the PDE given by:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),$$

becomes:

$$G'\phi = \frac{kG}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right).$$

Separating variables gives

$$\frac{G'}{kG} = \frac{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)}{r\phi} = -\lambda.$$

It follows that the two ODEs are

$$G' + k\lambda G = 0 \qquad \text{and} \qquad \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0.$$